Algebraic Actions Homework

Lecture I :

1. Show that an automorphism of a compact abelian gp preserves Haar measure.

2. Find an $A \in GL(n, \mathbb{Z})$ with an eigenvalue of absolute value 1 that is <u>not</u> a root of unity. What is the smallest possible n?

3. If $A \in GL(n, \mathbb{Z})$ and $\hat{A}$ on $\mathbb{Z}^n$ has a finite nonzero orbit, show that A is not ergodic.

4. Show that if $(x-\lambda_1)\cdots(x-\lambda_r) \in \mathbb{Z}[x]$ and each $|\lambda_j| = 1$, then each λ_j is a root of unity. [Hint: consider the polynomials
$$f_n(x) = \prod_{j=1}^{r} (x - \lambda_j^n) \,]$$

Lecture II

1. Prove that $\binom{2^n}{k} \equiv 0 \pmod 2$ for $1 \le k \le 2^n - 1$.

2. Show that every point in the classical Ledrappier example has unique canonical coordinates

3. Check that $\mathbb{F}_p((t))$ is a locally compact field, and that
$$\mathbb{F}_p[[t]] = \left\{ \sum_{n=0}^{\infty} c_n t^n : c_n \in \mathbb{F}_p \right\}$$
is a compact subring of $\mathbb{F}_p((t))$

4. Show that the classical Ledrappier example is mixing via orbits on dual group.

Lecture III

1. Prove that $\mathbb{Z}\mathbb{H}$ is a domain, i.e. if $f \cdot g = 0$, then $f = 0$ or $g = 0$.

2. Find a right Følner sequence for $\mathbb{H}$ that is <u>not</u> a left Følner sequence

3. Find $(x^2 - x - 1)^{-1}$ in $\ell^1(\mathbb{Z})$
   [Hint: partial fractions]
   Can you find a $g(x) \in \mathbb{Z}[x^{\pm}]$ such that $(x^2 - x - 1)\, g(x)$ is lopsided?

4. $\Gamma = \mathbb{Z}$, $f(x) = x^2 - x - 1$, $T = \rho_f \in \mathcal{B}(\ell^2(\mathbb{Z}))$.
   (a) Compute $\|T\|_{\mathcal{B}(\ell^2(\mathbb{Z}))}$.
   (b) Show that $\mu_{|T|}$ is given by a density ϕ on $[0, \|T\|]$, i.e.
   $$d\mu_{|T|}(t) = \phi(t)\, dt$$
   (c) Plot $\phi(t)$ as carefully as you can.

Lecture IV

1. Let $f = 1 + x + y \in \mathbb{Z}\mathbb{F}_2$.
   Compute $\operatorname{tr}\left[(f^* f)^m\right]$
   for $m = 1, 2, 3$, and more if you can.

2. Demonstrate residual finiteness of $\mathbb{F}_2$ by finding a sequence $\{\Gamma_n\}_1^\infty$ of finite-index normal subgroups $\Gamma_1 \supset \Gamma_2 \supset \cdots$ with $\bigcap_{n=1}^\infty \Gamma_n = \{e\}$.

3. Let $f = 3 - x - y$. Use geometric series to expand $\frac{1}{f} = w = \sum_{\gamma \in \mathbb{F}_2} w_\gamma \cdot \gamma$, and show that w is a homoclinic point for α_f, indeed that $\sum_{\gamma \in \mathbb{F}_2} |w_\gamma| = 1$.