Lecture Notes on Algebraic Actions

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Lecture I: Algebraic $\mathbb{Z}$-actions

¿ What is an algebraic action?

Γ: discrete countable gp

(Main examples: $\Gamma = \mathbb{Z}, \mathbb{Z}^d$,
$\mathbb{H}$ = discrete Heisenberg gp,
$\mathbb{F}_2$ = free gp on two generators)

X: compact abelian gp

μ: Haar measure on X, $\mu(X) = 1$

$\alpha : \Gamma \longrightarrow \mathrm{aut}(X)$: alg Γ-action

Halmos (1943): α automatically preserves μ.

¿Why?

- Endless supply of concrete, interesting examples with wide variety of behavior
- $h_{top} = h_\mu$
- Most interesting case have finite descriptions using integer data, so algorithmic questions are reasonable
- Lots of machinery available: duality, commutative algebra, algebraic geometry, ...
- Can get very explicit answers to dynamical questions
- Significant interactions with other parts of math (diophantine approximation, von Neumann algebras, ...)
- Testing ground for sofic entropy (where few examples exist)

Beginning example (simple, but very important!):

$\Gamma = \mathbb{Z}$

$X = \mathbb{T}^2 \quad (\mathbb{T} = \mathbb{R}/\mathbb{Z})$

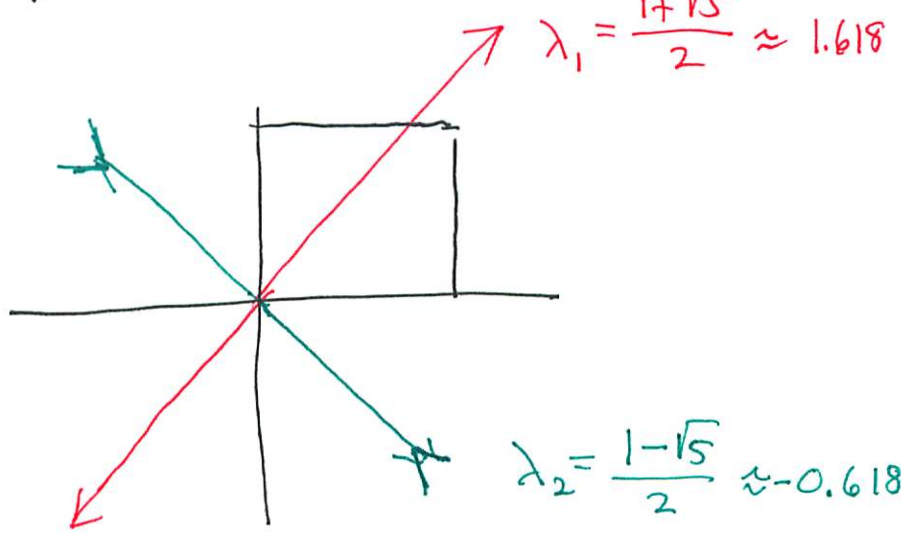
$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ acts on $\mathbb{T}^2$

$\alpha : \mathbb{Z} \longrightarrow \mathrm{aut}(\mathbb{T}^2); \quad \alpha(n) = A^n$

Geometry:

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\tilde{A}"] \arrow[d, "(\mathrm{mod}\ 1)"'] & \mathbb{R}^2 \arrow[d] \\ \mathbb{T}^2 \arrow[r, "A"] & \mathbb{T}^2 \end{tikzcd}$$

"linearization of A"



$|\lambda_1 \lambda_2| = 1 \Rightarrow A$ preserves μ.

¿ Is A ergodic? $E \subset \mathbb{T}^2$, $A(E) = E \overset{?}{\Rightarrow} \mu(E) = 0$ or 1?

Prove: mixing: $\mu(E \cap A^n F) \to \mu(E)\mu(F)$

<u>Duality</u>: $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\} = e^{2\pi i \mathbb{T}}$

Character of $\mathbb{T}$: $\chi : \mathbb{T} \to \mathbb{S}$ gp hm

$n = \#$ times χ wraps $\mathbb{T}$ around $\mathbb{S}$

$\Rightarrow \chi = \chi_n$, $\chi_n(t) = e^{2\pi i n t}$

$\hat{\mathbb{T}} = \{\text{all } \chi\} = \{\chi_n : n \in \mathbb{Z}\} \cong \mathbb{Z}$ (dual gp)

$\{\chi_n\}$ is an orthonormal basis for $L^2(\mathbb{T})$

$f \in L^2(\mathbb{T}) \longrightarrow$ Fourier coeff's $\hat{f}(n) = \langle f, \chi_n \rangle$

$= \int_0^1 f(t) e^{-2\pi i n t}\, dt$, $\{\hat{f}(n)\} \in \ell^2(\mathbb{Z})$

$L^2(\mathbb{T}) \overset{\wedge}{\cong} \ell^2(\mathbb{Z})$ Hilbert space isomorphism

$\widehat{\mathbb{T}^2} = \{\chi_{(m,n)} : (m,n) \in \mathbb{Z}^2\}$ $\chi_{(m,n)}(s,t) = e^{2\pi i (ms + nt)}$

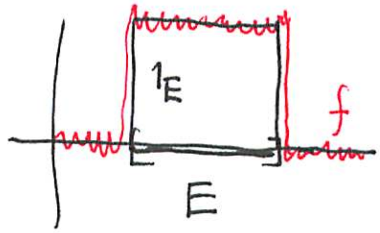
A on $\mathbb{T}^2$ dualizes to $\hat{A}$ on $\mathbb{Z}^2$

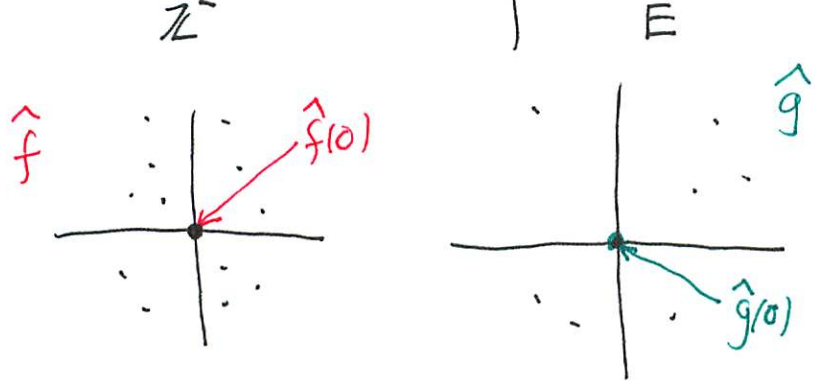
$\hat{A} = A^T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Back to mixing:

$$\mu(E \cap A^n F) = \int_{\mathbb{T}^2} \overbrace{1_E \cdot (1_F \circ A^{-n})}^{0^{\text{th}} \text{ Fourier coeff}} \, d\mu$$

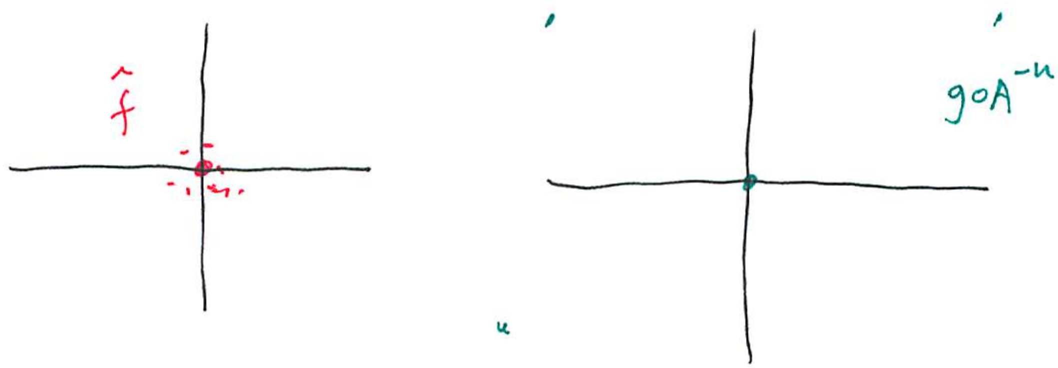
Approximate $1_E, 1_F$ by trig polys f, g





Analyse 0^{th} Fourier coeff of $f \cdot (g \circ A^{-n})$

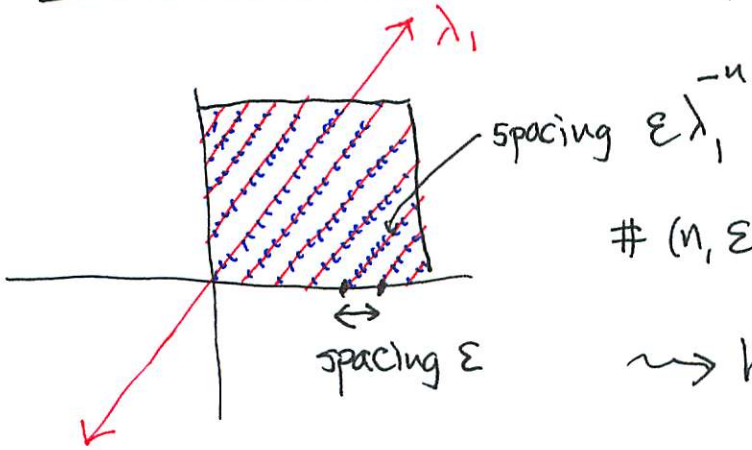
Essential point: If $\chi \neq \chi_{(0,0)}$, then $\hat{A}^n \chi \to \infty$ (all nonzero orbits of $\hat{A}$ on $\mathbb{Z}^2$ infinite)



$\therefore \int f \cdot (g \circ A^{-n}) \, d\mu \to \hat{f}(0) \hat{g}(0) = (\int f)(\int g)$

Now approx 1_E and 1_F by trig polys.

Top. entropy : ① (n, ε)-spanning sets



$\#\,(n,\varepsilon)$-spanning set $\sim \frac{1}{\varepsilon^2}\lambda_1^n$

$\rightsquigarrow h_{top}(A) = \log \lambda_1$

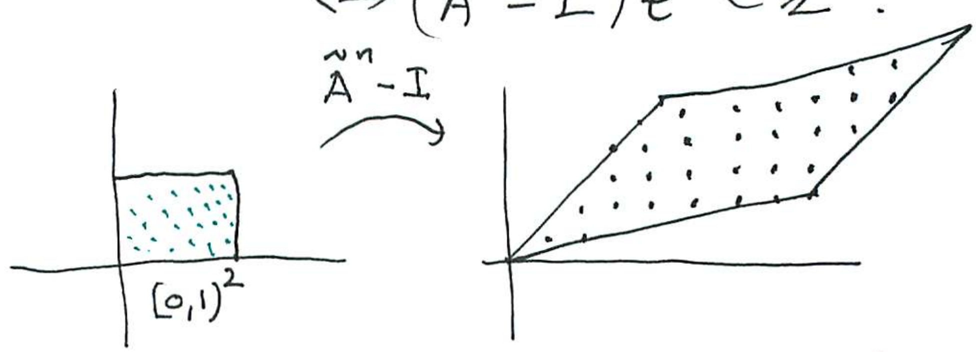
② Periodic points: $p_n = p_n(A) = \#$ n-periodic pts

$$\frac{1}{n}\log p_n(A) \to h_{top}(A).$$

Let $t \in \mathbb{T}^2$ and $\tilde{t} \in [0,1)^2$ its "lift" to $\mathbb{R}^2$.

$$A^n t = t \pmod 1 \iff (A^n - I)t = 0 \pmod 1$$

$$\iff (\tilde{A}^n - I)\tilde{t} \in \mathbb{Z}^2.$$



$$p_n(A) = \#\text{ lattice pts in } (\tilde{A}^n - I)[0,1)^2$$

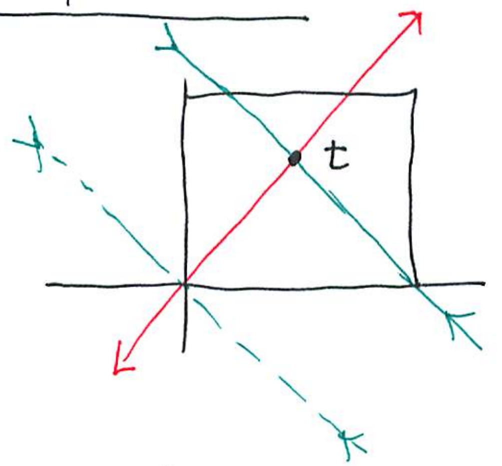
$$= |\det(\tilde{A}^n - I)|$$

$$= |(\lambda_1^n - 1)(\lambda_2^n - 1)| \sim \lambda_1^n \rightsquigarrow h = \log\lambda_1$$

[First appearance of det to compute entropy]

More dynamical properties of $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$:

- <u>Expansive</u>: $\exists\, \delta > 0$ s.t. if $\mathrm{dist}(A^n t, A^n u) < \delta$ for all $n \in \mathbb{Z}$, then $t = u$.
  - A version of hyperbolicity
  - Important "finiteness" condition
- <u>Homoclinic points</u>: $A^n t \to 0$ as $|n| \to \infty$

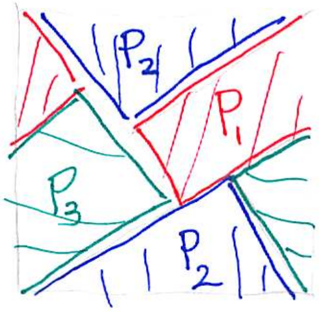


$\Delta(A) = \{\text{hc pts}\}$

countable dense subgp

  - Ergodic theorem $\Rightarrow$ for a.e. $t \in \mathbb{T}^2$, $\{A^n t\}$ is uniformly distributed; h.c. pts behave very differently
  - Hc pts are a powerful general tool for alg. actions
  - Discovered by Poincaré when he found a <u>horrible</u> mistake in a prize essay he wrote, journals recalled!

- Markov partitions (Adler, Weiss 1968)



$(A, \mathbb{T}^2) \cong$ Markov shift with transition matrix

$$B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$h = \log \lambda_B$ (λ_B = spectral radius of B)

- Like decimal expansions, "digits"
- The start of "symbolic dynamics", combinatorial description of dynamical systems
- Powerful tool in smooth dynamics

- Bernoulli: $\exists$ partition $\mathcal{P} = \{Q_1, Q_2, Q_3\}$ of $\mathbb{T}^2$ into measurable sets st
  - $\{A^n \mathcal{P}\}$ are jointly independent:
  $$\mu\Big(\bigcap_{j=1}^{r} A^j Q_{i_j}\Big) = \prod_{j=1}^{r} \mu(Q_{i_j})$$
  - $\bigvee_{-\infty}^{\infty} A^n \mathcal{P}$ is all measurable sets

But No one has ever "written down" such a partition!

Toral autos: $A \in GL(r, \mathbb{Z}) \rightsquigarrow A \in \operatorname{aut}(\mathbb{T}^r)$

eigenvalues $\lambda_1, \dots, \lambda_r$

Property	Characterization
Ergodic	No λ_j is a root of unity
Expansive	$\lvert\lambda_j\rvert \neq 1$ all j
Entropy	$h = \sum_j \log^+ \lvert\lambda_j\rvert = \sum_{\lvert\lambda_j\rvert>1} \log \lvert\lambda_j\rvert$
# n-per pts	$\left\lvert \prod_{j=1}^{r} (\lambda_j^n - 1) \right\rvert$ (provided erg)
Bernoulli	Ergodic

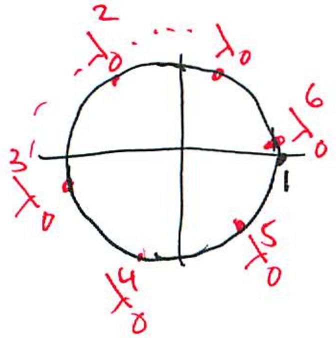
Growth rate of periodic pts:

Does $\frac{1}{n} \log p_n(A) = \frac{1}{n} \sum_{j=1}^{r} \log \lvert\lambda_j^n - 1\rvert \xrightarrow{?} h$

Z If some $\lvert\lambda_{j_0}\rvert = 1$ (but not a root of unity)

Then $\lvert\lambda_{j_0}^k - 1\rvert$ can get very small,

then $\log \lvert\lambda_{j_0}^k - 1\rvert \sim -\infty$.



A deep theorem of Gelfond: $\forall \varepsilon > 0$,

$\lvert\lambda_{j_0}^k - 1\rvert \gg e^{-\varepsilon k}$ as $k \to \infty$

and convergence holds.

Reformulation of basic example

$\mathbb{T}^{\mathbb{Z}}$: $\quad t = (\dots, t_{-1}, t_0, t_1, \dots)$ $\quad (t_j \in \mathbb{T})$

α: $\leftarrow$ shift

$$\mathbb{T}^{\mathbb{Z}} \supset X = \{ t \in \mathbb{T}^{\mathbb{Z}} : \overset{\overset{-1}{\cdot}}{t_{k+2}} - \overset{\overset{-1}{\cdot}}{t_{k+1}} - \overset{\overset{1}{\cdot}}{t_k} = 0 \ \forall k \}.$$

Integral condition

Claim $(X, \alpha) \cong (\mathbb{T}^2, A)$

$$\begin{array}{ccc} (\dots, t_{-1}, \underbrace{\dot{t}_0, t_1}, \dots) & \xrightarrow{\alpha} & (\dots, t_0, \underbrace{\dot{t}_1, t_2}, \dots) \\ \downarrow & & \downarrow \\ \mathbb{T}^2 \ni \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} & \xrightarrow{A} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t_0 \\ t_1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \end{array}$$

Duality: $\widehat{\mathbb{T}^{\mathbb{Z}}} \cong \bigoplus_{\mathbb{Z}} \mathbb{Z} \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{Z} x^n = \mathbb{Z}[x^{\pm}]$

$$\alpha = \leftarrow \quad \overset{\wedge}{\longleftrightarrow} \quad \times x$$

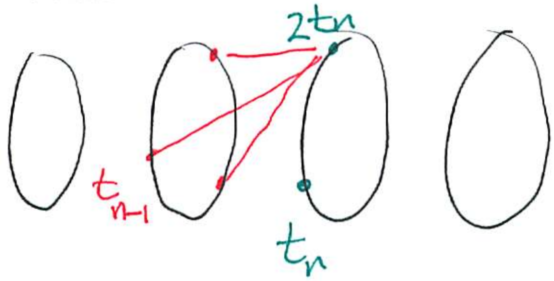
$$\begin{array}{ccc} \mathbb{T}^{\mathbb{Z}} & \overset{\wedge}{\longleftrightarrow} & \mathbb{Z}[x^{\pm}] \\ \cup & & \downarrow \\ X & \overset{\wedge}{\longleftrightarrow} & \mathbb{Z}[x^{\pm}] / \langle x^2 - x - 1 \rangle \end{array}$$

Can do this for any polynomial $f(x) \in \mathbb{Z}[x^{\pm}]$

$$(X_f, \alpha_f) \overset{\wedge}{\longleftrightarrow} (\mathbb{Z}[x^{\pm}]/\langle f(x)\rangle, \times x)$$

This α_f is called a <u>principal algebraic action</u> of $\mathbb{Z}$.

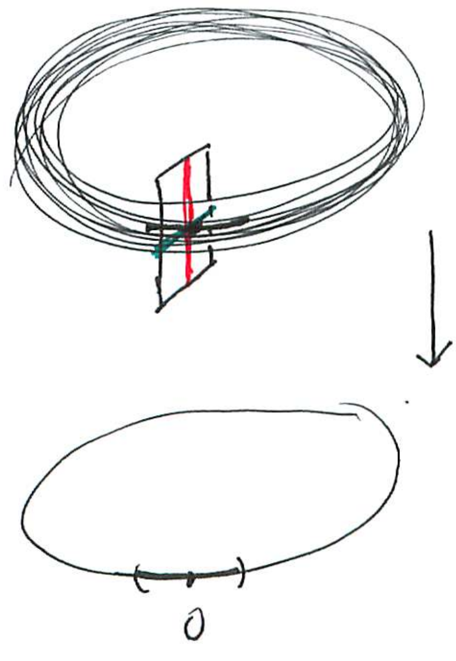
<u>Example</u>: $f(x) = 2x - 3$



$$\mathbb{Z}[x^{\pm}]/\langle 2x-3\rangle \cong \mathbb{Z}[\tfrac{1}{6}], \times \tfrac{3}{2}$$

$$3t_{n-1} = 2t_n$$

¿what does X_f look like?

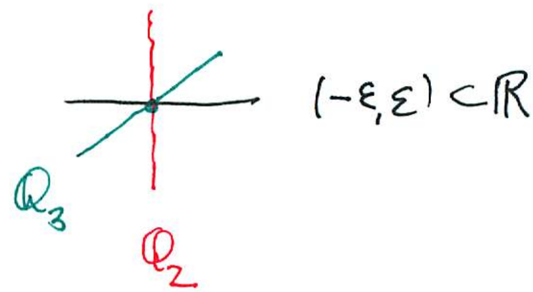


$$\mathbb{Z}[\tfrac{1}{6}] \longleftrightarrow \mathbb{Z}$$

$\mathbb{Q}_p$ = p-adic field

X_f locally looks like

$(-\varepsilon, \varepsilon) \times \mathbb{Q}_2 \times \mathbb{Q}_3$



and α_f is $\times \frac{3}{2}$ in each coordinate

		entropy:
$\times \frac{3}{2}$ on $\mathbb{R}$	size $\left\|\frac{3}{2}\right\|_\infty = \frac{3}{2}$	$\log \frac{3}{2}$
$\times \frac{3}{2}$ on $\mathbb{Q}_2$	size $\left\|\frac{3}{2}\right\|_2 = 2$	$\log 2$
$\times \frac{3}{2}$ on $\mathbb{Q}_3$	size $\left\|\frac{3}{2}\right\|_3 = \frac{1}{3}$	0
		$\log 3 = h(\alpha_f)$.

Here there is geometric expansion ($\mathbb{R}$) as well as arithmetic expansion ($\mathbb{Q}_p$) and "p-adic eigen values" play an essential role. "Adelic analysis" and "local/global principle".

<u>Mahler measure</u>: $0 \neq f(x) \in \mathbb{Z}[x^{\pm}]$,

$$m(f) := \text{log Mahler measure of } f$$
$$= \int_0^1 \log |f(e^{2\pi i s})|\, ds = \int_{\mathbb{S}} \log |f|$$

$$M(f) = \text{Mahler measure of } f = e^{m(f)}$$

<u>Rf</u>: $M(fg) = M(f) \cdot M(g)$

<u>Jensen</u>: $m(x-\lambda) = \int_0^1 \log |e^{2\pi i s} - \lambda|\, ds$
$$= \log^+ |\lambda|.$$

<u>Thm</u> (Yuzvinskii): For $0 \neq f \in \mathbb{Z}[x^{\pm}]$
$$h(\alpha_f) = m(f) = \log M(f)$$

If $f(x) = c_r x^r + \ldots + c_0 = c_r \prod_{j=1}^{r} (x - \lambda_j)$,

$$m(f) = \underbrace{\log |c_r|}_{\text{sum of all } p\text{-adic contributions}} + \underbrace{\sum_{j=1}^{r} \log^+ |\lambda_j|}_{\text{geometric contribution}}$$

<u>Kronecker's Thm</u>: If $f(x) = \prod_{j=1}^{r} (x-\lambda_j) \in \mathbb{Z}[x]$ and $|\lambda_j| = 1$ all j, Then all λ_j are roots of unity. ($\sqrt{1}$).

<u>Hence</u>: $h(\alpha_f) = m(f) = 0$ iff all roots of f are $\sqrt{1}$.

<u>Riemann sums</u>: $\Omega_n = \{ e^{2\pi i k/n} : 0 \le k < n \}$

Ω_n 

$$\frac{1}{n} \sum_{\zeta \in \Omega_n} \log |f(\zeta)| \xrightarrow{?} \int_{\mathbb{T}} \log |f| \quad ?$$

True, but need Gelfond's result.

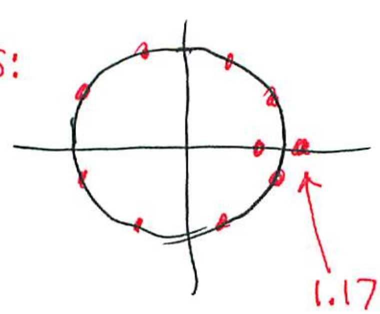
<u>Lehmer problem</u>: Is There an $f \in \mathbb{Z}[x]$ st

$$0 < m(f) < 0.1 \quad ?$$

Best known:

$$f(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

Roots:



$m(f) \approx 0.157\ldots$