Lecture II: Algebraic $\mathbb{Z}^d$-actions

Motivating example (Ledrappier (1978), ":. for $\mathbb{Z}/2\mathbb{Z}$")

Start with $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$, compact gp, $\mathbb{Z}^2$-action gen by $\leftarrow$, $\downarrow$.

 $\to 0$

$$(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} \supset X := \{ t \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : t_{k,\ell} + t_{k+1,\ell} + t_{k,\ell+1} = 0 \ \forall k,\ell \}$$

Typical pt in X

0 0 1 1 0 1 0 1 0 1 1 1
1 1 1 0 1 1 0 0 1 1 0 1 0
... 0 1 0 1 1 0 1 1 1 0 1 1 0 0 ...
0 0 1 1 0 1 1 0 1 0 0 1 0
0 0 0 1 0 0 1 0 0 1 1 1 0
⋮

X is $\mathbb{Z}^2$-shift invariant (:. everywhere) and a compact subgroup (:. is additive)

$(X, \alpha = \langle \leftarrow, \downarrow \rangle)$ "classical Ledrappier"

¿Why? r-fold mixing problem:

$T: (Y,\nu) \to (Y,\nu)$ meas. preserving

2-mixing: $\nu(E_1 \cap T^n E_2) \xrightarrow{n\to\infty} \nu(E_1)\nu(E_2)$

3-mixing: $\nu(E_1 \cap T^n E_2 \cap T^{n+m} E_3) \to \prod_1^3 \nu(E_j)$

as $n, m \to \infty$

¿ 2-mix $\Rightarrow$ 3-mix ? Still open!!

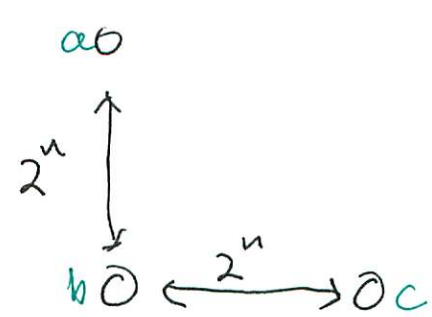
Classical Ledrappier is $\mathbb{Z}^2$-action That is 2-mix but <u>not</u> 3-mix.

$$
\begin{array}{ccccc}
1 & & & & \\
1 & 1 & & & \\
1 & 2 & 1 & & \\
1 & 3 & 3 & 1 & \\
1 & \cancel{4}\to 0 & \cancel{6}\to 0 & \cancel{4}\to 0 & 1
\end{array}
$$

$\binom{2^n}{k} \equiv 0 \pmod 2$

for all $1 \le k \le 2^n - 1$

So have long-range 3-correlations



$a+b+c \equiv 0 \pmod 2$

<u>Deeper structure</u>: There are exactly three "flavors" of locally compact fields

- $\mathbb{R}$ and $\mathbb{C}$ [char $=0$, archimedian]
- $\mathbb{Q}_p$ and $\mathbb{Q}_p(\theta)$ [char $=0$, non-arch]
- $\mathbb{F}_p((t))$ + finite ext's [char $=p$, non-arch]

$$\mathbb{F}_2((t)) = \{ a_{-n}t^{-n} + a_{-n+1}t^{-n+1} + \dots + a_0 + a_1 t + \dots : a_j \in \mathbb{F}_2 \text{ all } j \}$$

How to invert? Geometric series:

$$\frac{1}{1-u} = 1 + u + u^2 + \dots$$

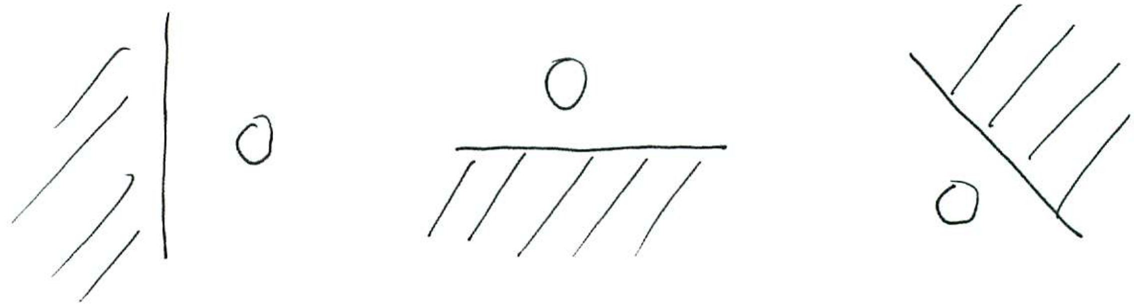
$$\frac{1}{1 - a_1 t - a_2 t^2 - \dots} = 1 + (a_1 t + a_2 t^2 + \dots) + (a_1 t + a_2 t^2 + \dots)^2 + (a_1 t + a_2 t^2 + \dots)^3 + \dots$$

(¿ convergent?)

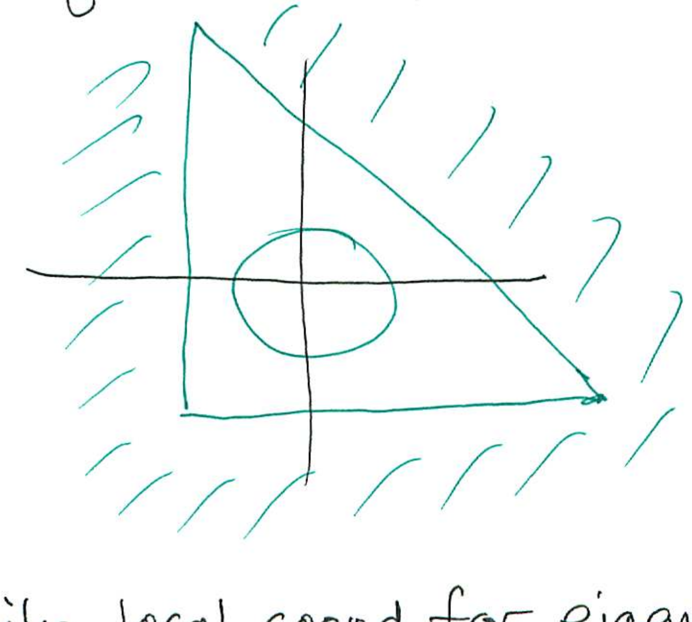
Topology induced from $\mathbb{F}_p^{\mathbb{Z}}$.

Recall the example $f(x) = 2x - 3$, "locally" like $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$, so near 0 there are three "coordinates".

Classical Ledrappier is locally a product of 3 copies of $\mathbb{F}_2((t))$:

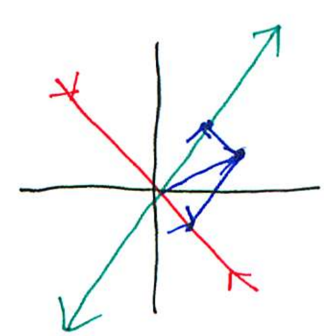


Any pt in X close to 0 is the unique sum of three coord pts



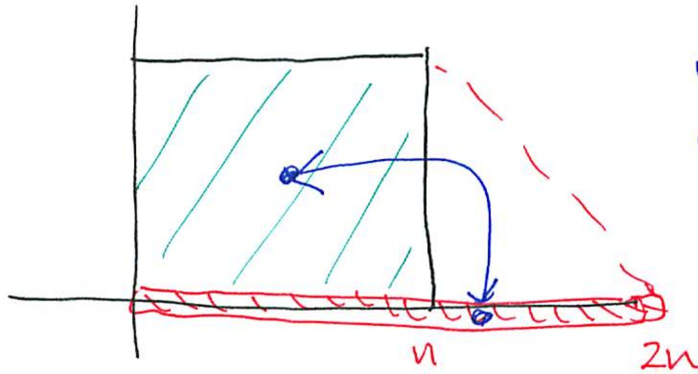
= sum of three pts as above.

Like local coord for eigensp:



<u>Entropy</u>: Look at $n \times n$ "windows"

$$\frac{1}{n^2}\left(\log \# \text{ patterns in } n\times n \text{ window}\right) \longrightarrow h$$



each determine the other

$$\frac{1}{n^2} \log 2^{2n} = \frac{2n}{n^2} \log 2 \to 0 \Rightarrow h = 0$$

Not the end of the story: There are 1-dimensional directional entropies.

<u>Rk</u>: $\{0, \frac{1}{2}\} \subset \mathbb{T}$ gp isom to $\mathbb{Z}/2\mathbb{Z}$.

So there is a version of classical Ledrappier using $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}\}$, and same argument shows <u>all</u> of these have $h=0$.

Let "$n \to \infty$", and use $\mathbb{T}$ in each coordinate, together with $\therefore$

What is h?

$$X = \{ t \in \mathbb{T}^{\mathbb{Z}^2} : t_{k,\ell} + t_{k+1,\ell} + t_{k,\ell+1} = 0 \ \forall k,\ell \}$$

$\alpha = \langle \leftarrow, \downarrow \rangle$. Duality:

$$\begin{array}{ccc}
\mathbb{T}^{\mathbb{Z}^2} & \overset{\wedge}{\longleftrightarrow} & \bigoplus_{\mathbb{Z}^2} \mathbb{Z} = \bigoplus_{k,\ell} \mathbb{Z} x^k y^\ell = \mathbb{Z}[x^{\pm}, y^{\pm}] \\
\Big\uparrow & & \Big\downarrow \\
X & \longleftrightarrow & \mathbb{Z}[x^{\pm}, y^{\pm}] / \langle 1 + x + y \rangle \\
\alpha & \longleftrightarrow & \langle \cdot x, \cdot y \rangle
\end{array}$$

Log Mahler measure of $f(x,y) \in \mathbb{Z}[x^{\pm}, y^{\pm}]$:

$$m(f) = \int_0^1 \int_0^1 \log |f(e^{2\pi i s_1}, e^{2\pi i s_2})| \, ds_1 \, ds_2$$

$$= \int_{\mathbb{S}^2} \log |f|$$

<u>Thm</u> (L-Schmidt-Ward):

$$h(\because \text{ for } \mathbb{T}) = m(1+x+y) \simeq 0.3230$$

Build up a plausibility argument for this.

Initial example: $f(x)=x^2-x-1$. $f^*(x)=x^{-2}-x^{-1}-1$.

$$\mathbb{T}^{\mathbb{Z}} \ni t \leftrightarrow \sum_{-\infty}^{\infty} t_n x^n$$

$$\begin{aligned}
\rho_f(t) &= t\cdot f^*(x)\\
&= (\dots + t_0 + t_1 x + t_2 x^2 + t_3 x^3 + \dots)(x^{-2}-x^{-1}-1)\\
&= \dots + (t_2-t_1-t_0) + (t_3-t_2-t_1)x + \dots
\end{aligned}$$

$$t \in X_f \iff \rho_f(t) = 0.$$

Compute p_n = # pts of period n.

Start with all period n pts, cut down to X_f

$$\begin{tikzcd}
{[0,1)^{\mathbb{Z}/n\mathbb{Z}} \ni \tilde{t} \in \mathbb{R}^{\mathbb{Z}/n\mathbb{Z}}} \arrow[r, "\tilde{\rho}_f^{(n)}"] \arrow[d] & \mathbb{R}^{\mathbb{Z}/n\mathbb{Z}} \arrow[d] \\
t \in \mathbb{T}^{\mathbb{Z}/n\mathbb{Z}} \arrow[r, "\rho_f^{(n)}"] & \mathbb{T}^{\mathbb{Z}/n\mathbb{Z}}
\end{tikzcd}$$

$$t \in X_f \iff \tilde{\rho}_f^{(n)}(\tilde{t}) \in \mathbb{Z}^{\mathbb{Z}/n\mathbb{Z}}$$

$$p_n = |\det \tilde{\rho}_f^{(n)}|$$

But can diagonalize shift:

$\Omega_n = n^{th}$ roots of unity

$$\zeta \in \Omega_n \rightsquigarrow v_\zeta = [1 \ \zeta \ \zeta^2 \ \dots \ \zeta^{n-1}] \in \mathbb{C}^{\mathbb{Z}/n\mathbb{Z}}$$

$$\rho_x(v_\zeta) = \zeta \cdot v_\zeta$$

$$\rho_f(v_\zeta) = \rho_{x^2-x-1}(v_\zeta) = (\zeta^2-\zeta-1)v_\zeta$$

$$\det \tilde{\rho}_f^{(n)} = \prod_{\zeta\in\Omega}(\zeta^2-\zeta-1) = \prod_{\zeta\in\Omega} f(\zeta)$$

$$\frac{1}{n}\log P_n = \frac{1}{n}\log|\det \tilde{\rho}_f^{(n)}|$$

$$= \frac{1}{n}\log\prod_{\zeta\in\Omega_n}|f(\zeta)| \to \int_{\mathbb{S}}\log|f| = m(f)$$

$$= \log\lambda_1$$

Recall we also had $P_n = |\det(A^n - I)|$

Why different expressions? Different ways to create periodic points:

"Internal" $\longrightarrow |\det(A^n - I)|$

start with these

require periodic

"External" Start with all n-periodic pts, cut down to $X_f \longrightarrow \left|\prod_{\zeta\in\Omega_n} f(\zeta)\right|$

Back to Ledrappier for $\mathbb{T}$:

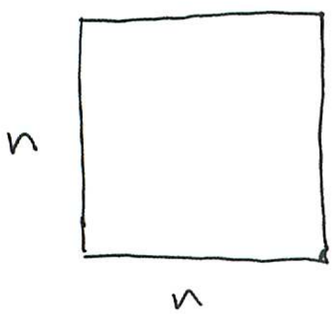
$$f(x,y) = 1 + x + y, \quad f^*(x,y) = 1 + x^{-1} + y^{-1}$$

$$t \in \mathbb{T}^{\mathbb{Z}^2} \longleftrightarrow t = \sum t_{k,\ell}\, x^k y^\ell$$

$$\rho_f(t) = t \cdot f^*$$

$$X_f = \ker \rho_f$$

Compute $n \times n$ periodic pts using "external"



$$(\xi, \eta) \in \Omega_n \times \Omega_n \longrightarrow v_{\xi,\eta} \in \mathbb{C}^{n \times n}$$

$$(v_{\xi,\eta})_{k,\ell} = \xi^k \eta^\ell$$

$$\tilde{\rho}_x(v_{\xi,\eta}) = \xi \cdot v_{\xi,\eta}$$

$$\tilde{\rho}_y(v_{\xi,\eta}) = \eta \cdot v_{\xi,\eta}$$

$$\tilde{\rho}_{1+x+y}(v_{\xi,\eta}) = (1 + \xi + \eta)\, v_{\xi,\eta}$$

$$= \underbrace{f(\xi,\eta)}_{n^2 \text{ eigenvalues}}\, v_{\xi,\eta}$$

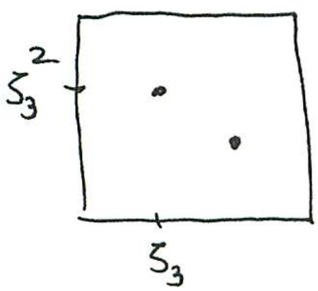
So

$$\det \tilde{\rho}_{1+x+y} = \prod_{(\xi,\eta) \in \Omega_n^2} f(\xi,\eta)$$

$$\frac{1}{n^2}\log \#\ n\times n \text{ per. pts} = \frac{1}{n^2}\sum_{(\xi,\eta)\in\Omega_n^2} \log|f(\xi,\eta)|$$

$$\to \int_{\mathbb{S}^2} \log|f| = m(f)$$

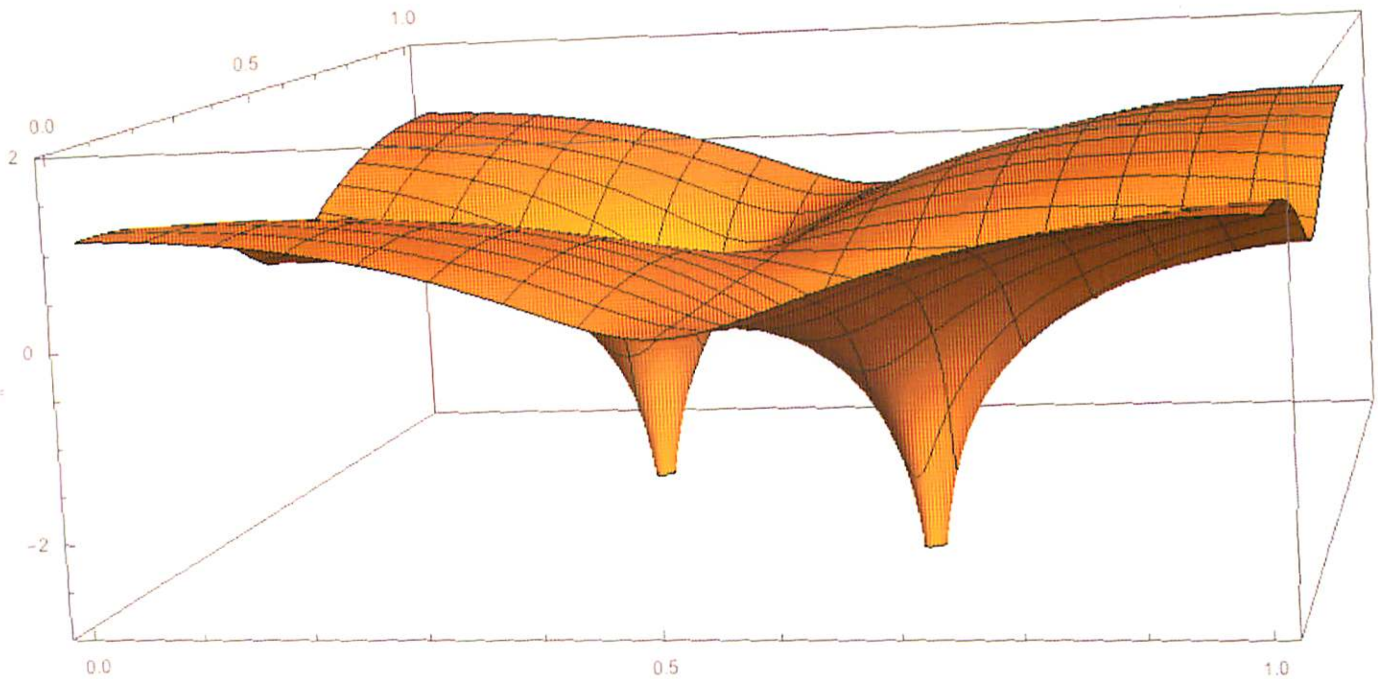
There is a serious problem. Let $\zeta_3 = e^{2\pi i/3}$, so $1+\zeta_3+\zeta_3^2=0$, and $f(\zeta_3,\zeta_3^2)=0=f(\zeta_3^2,\zeta_3)$!



So if $3|n$, two of the terms are $\log 0 = -\infty$. (graph of $\log|f|$ next page)

This is a recurring headache: various ways to deal with it.

- "Perturb" to something very similar, but which does not vanish
- Use the internal method and "integrate away" singularities
- Count periodic <u>components</u> instead of periodic points.



Graph of $\log|1+e^{2\pi i s_1}+e^{2\pi i s_2}|$

Principal $\mathbb{Z}^d$-actions

$$R_d = \mathbb{Z}[x_1^{\pm}, \dots, x_d^{\pm}]$$

$$f(x_1, \dots, x_d) \in R_d$$

$$\mathbb{T}^{\mathbb{Z}^d} \supset X_f = \ker \rho_f \overset{\wedge}{\longleftrightarrow} R_d/\langle f \rangle$$

$$\alpha_f \longleftrightarrow \langle \times x_1, \times x_2, \dots, \times x_d \rangle$$

Thm (L-S-W): $h(\alpha_f) = m(f) = \log M(f)$.

Lehmer's problem for R_d?

$$m(f(x, x^n)) \longrightarrow m(f(x,y))$$

so it's the same problem as for R_1.

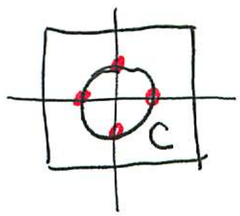
Zero Entropy: Generalize cyclotomic
$\Phi(x_1^{n_1} \cdots x_d^{n_d})$ where $\Phi(u)$ is cyclotomic

$m(f) = 0 \iff f$ is a product of generalized cyclotomics

Convergence of Riemann sums:

Let $f(x,y) = 3 - x - x^{-1} - y - y^{-1}$

$f = 0$ along a curve in $\mathbb{S}^2$, so $\log|f|$ has a curve of singularities.



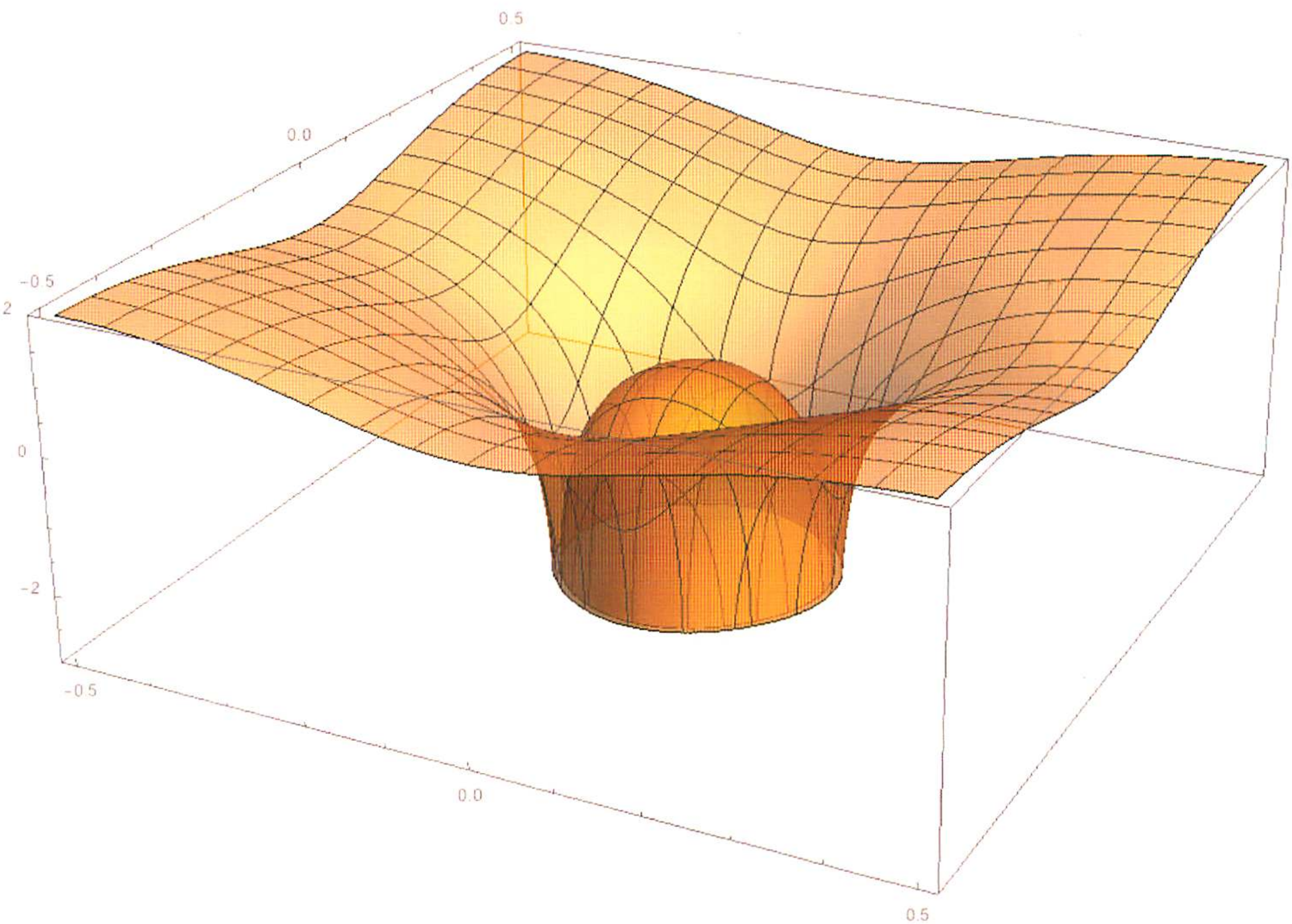
The only roots of unity on C are $(1, \zeta_6^{\pm 1})$ and $(\zeta_6^{\pm 1}, 1)$

Let $F \subset \mathbb{S}^2$ be a finite subgroup

omit the roots of unity where $f=0$.

Does $\displaystyle \frac{1}{|F|} \sum_{(\zeta,\eta)\in F}{}' \log|f(\zeta,\eta)| \overset{?}{\to} m(f)$

Very recent work shows this is true for "square" subgroups $\Omega_n \times \Omega_n$.
There are two proofs, both very hard, one using logic and O-minimal sets!
The question for general F's is still wide open.



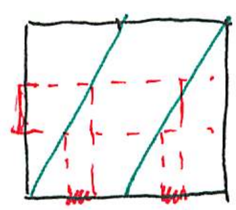
Graph of $\log|f|$ on $\mathbb{S}^2$, where

$f(x,y) = 3 - x - x^{-1} - y - y^{-1}$

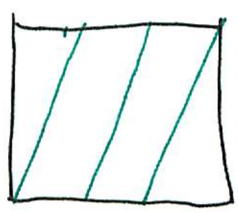
There is much more known about algebraic $\mathbb{Z}^d$-actions, and a fairly complete "dictionary", where dynamical properties can be translated into commutative algebra and algebraic geometry (and answered!).

But lest you think everything is known:

Consider $M_2, M_3 : \mathbb{T} \to \mathbb{T}$, $M_k(t) = kt$



M_2



M_3

Haar measure is preserved by both.
Some atomic measures too: $\frac{1}{5}\sum_{j=0}^{4} \delta_{j/5}$.
Are there any others?

<u>Furstenberg's problem</u> (or $\times 2, \times 3$)

Has been open for 50 years!!!