Lecture II: Algebraic $\mathbb{Z}^d$-actions

Motivating example (Ledrappier (1978), "for $\mathbb{Z}/2\mathbb{Z}$")

Start with $(\mathbb{Z}/2\mathbb{Z})^2$, compact gp, $\mathbb{Z}^2$-action

gen by $\leftarrow$, $\downarrow$.

$$(\mathbb{Z}/2\mathbb{Z})^2 \rightarrow X := \{ t \in (\mathbb{Z}/2\mathbb{Z})^2 : \quad t_{k,e} + t_{k+1,e} + t_{k,e+1} = 0 \quad \forall k, e \}$$

Typical pt in $X$

\[ \begin{array}{cccccccccccccc}
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
\end{array} \]

$X$ is $\mathbb{Z}^2$-shift invariant (\ldots everywhere) and a compact subgroup (\ldots is additive)

$(X, \alpha = \leftarrow, \downarrow)$ "classical Ledrappier"
Why? r-fold mixing problem:

\[ T: (Y, \nu) \to (Y, \nu) \text{ meas. preserving} \]

2-mixing: \( \nu(E_1 \cap T^n E_2) \to \sqrt{n} \nu(E_1) \nu(E_2) \)

3-mixing: \( \nu(E_1 \cap T^n E_2 \cap T^{n+m} E_3) \to \frac{\nu(E_1)}{\nu(E_1)} \)

as \( n, m \to \infty \)

¿ 2-mix \implies 3-mix ? Still open !!

Classical Ledrappier is \( \mathbb{Z}^2 \)-action That is 2-mix but not 3-mix.

\[
\begin{array}{cccc}
1 & 1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
\end{array}
\]

\( \binom{2^n}{k} \equiv 0 \pmod{2} \) for all \( 1 \leq k \leq 2^n - 1 \)

So have long-range 3-correlations

\[ a + b + c \equiv 0 \pmod{2} \]
Deeper structure: There are exactly three "flavors" of locally compact fields

- \( \mathbb{R} \) and \( \mathbb{C} \) \([\text{char}=0, \ \text{archimedean}]\)
- \( \mathbb{Q}_p \) and \( \mathbb{Q}_p(\Theta) \) \([\text{char}=0, \ \text{non-arch}]\)
- \( \mathbb{F}_p((t)) \) + finite ext's \([\text{char}=p, \ \text{non-arch}]\)

\( \mathbb{F}_2((t)) = \left\{ a_{-n}t^{-n} + a_{-n+1}t^{-n+1} + \ldots + a_0 + a_1 t + \ldots : \right. \\
\left. a_j \in \mathbb{F}_2 \ \text{all} \ j \right\} \)

How to invert? Geometric series:
\[
\frac{1}{1-u} = 1 + u + u^2 + \ldots \\
\frac{1}{1-a_1 t - a_2 t^2 - \ldots} = 1 + (a_1 t + a_2 t^2 + \ldots)^2 \\
+ (a_1 t + a_2 t^2 + \ldots)^3 \\
+ \ldots \quad (\text{is convergent?})
\]

Topology induced from \( \mathbb{F}_p^2 \).
Recall the example $f(x) = 2x - 3$, "locally" like $\mathbb{R} \times \mathbb{Q} \times \mathbb{Q}$, so near 0 there are three "coordinates".

Classical Ledvappier is locally a product of 3 copies of $\mathbb{F}_2((t))$.

Any pt in $x$ close to 0 is the unique sum of three coord pts $= \text{sum of Three pts as above.}$

Like local coord for eigensp.
Entropy: Look at nxn "windows"

\[ \frac{1}{n^2} \log \text{# patterns in nxn window} \rightarrow h \]

each determine the other

\[ \frac{1}{n^2} \log 2^{2n} = \frac{2n}{n^2} \log 2 \rightarrow 0 \Rightarrow h = 0 \]

Not the end of the story: There are 1-dimensional directional entropies.

RK: \[ \{0, \frac{1}{2}\} \subset \mathbb{T} \text{ gp isom to } \mathbb{Z}/2\mathbb{Z}. \]

So there is a version of classical Ledrappier using \[ \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\} \], and same argument shows all of these have \( h = 0 \).
Let \( n \to \infty \), and use \( \mathbb{T} \) in each coordinate, together with \( \ldots \).
What is \( h \)?
\[ X = \{ t \in \mathbb{P}^n \mathbb{Z}^2 : t_{k,e} + t_{k+1,e} + t_{k,e+1} = 0 \ \forall k,e \} \]

\[ \alpha = \langle \cdot, \cdot \rangle. \text{ Duality:} \]

\[ \bigoplus \mathbb{Z}^2 \xrightarrow{\sim} \bigoplus \mathbb{Z} = \bigoplus \mathbb{Z} x_k^e = \mathbb{Z}[x^\pm, y^\pm] \]

\[ X \xrightarrow{\sim} \mathbb{Z}[x^\pm, y^\pm]/\langle 1 + x + y \rangle \]

\[ \alpha \xrightarrow{\sim} \langle \cdot x, \cdot y \rangle \]

Log Mahler measure of \( f(x,y) \in \mathbb{Z}[x^\pm, y^\pm] \):

\[ m(f) = \int_0^1 \int_0^1 \log |f(e^{2\pi i s_1}, e^{2\pi i s_2})| \, ds_1 \, ds_2 \]

\[ = \int_{S^1} \log |f| \]

**Thm (L-Schmidt-Ward):**

\[ h(\cdot : \cdot \text{ for } \mathbb{P}) = m(1 + x + y)^2 \leq 0.3230 \]
Build up a plausibility argument for $T_n$.

Initial example: $f(x) = x^2 - x - 1$. $f^*(x) = x^2 - x - 1$.

\[ \prod_{x \in \mathbb{Z}} e^{t} \iff \sum_{n=-\infty}^{\infty} t_n x^n \]

\[ p_f(t) = t \cdot f^*(x) \]

\[ = (\ldots + t_0 + t_1 x + t_2 x^2 + t_3 x^3 + \ldots) (x^2 - x - 1) \]

\[ = \ldots + (t_2 - t_1 - t_0) + (t_3 - t_2 - t_1) x + \ldots \]

\[ t \in X_f \iff p_f(t) = 0. \]

Compute $p_n = \# pts$ of period $n$.

Start with all period $n$ pts, cut down to $X_f$.

\[ [0,1) \in \mathbb{R} \xrightarrow{\sim} \mathbb{Z} \xrightarrow{p_f^{(n)}} \mathbb{Z} \]

\[ t \in \mathbb{Z}/\mathbb{Z} \xrightarrow{p_f^{(n)}} \mathbb{Z}/\mathbb{Z} \]

\[ t \in X_f \iff \tilde{p}_f^{(n)}(\tilde{t}) \in \mathbb{Z} \]

\[ p_n = |\det \tilde{p}_f^{(n)}| \]
But can diagonalize shift:
\[ \mathbb{Z}_n \cong \mathbb{Z}^n \]
\[ \mathbb{Z}_n \rightarrow \mathbb{Z}^n = [1 \ 1 \ 5^2 \ \ldots \ 5^{n-1}] \in \mathbb{Z}^{n \times n} \]
\[ p_x(\mathbf{v}_5) = \mathbf{v}_5 \]
\[ p_f(\mathbf{v}_5) = p_{x^5-x-1}(\mathbf{v}_5) = (5^2 - 5 - 1) \mathbf{v}_5 \]
\[ \text{det } \hat{p}_f = \prod_{\mathbf{v}_5} (5^2 - 5 - 1) = | \prod_{\mathbf{v}_5} f(5) | \]
\[ \mathbf{v}_5 \in \mathbb{Z}_n \]

\[ \frac{1}{n} \log p_n = \frac{1}{n} \log | \text{det } \hat{p}_f | \]
\[ = \frac{1}{n} \log | \prod_{\mathbf{v}_5} |f(5)| | \rightarrow \sum_{\mathbf{v}_5} \log |f| = m(f) \]
\[ = \log \lambda_1 \]

Recall we also had \[ p_n = | \text{det } (A^n - I) | \]

Why different expressions? Different ways to create periodic points:

"Internal" \[ \begin{array}{c}
\text{start with these} \quad \rightarrow | \text{det } (A^n - I) | \\
\text{require periodic}
\end{array} \]

"External" \[ \begin{array}{c}
\text{start with all } n \text{-periodic pts} \\
\text{cut down to } x_f \rightarrow | \prod_{\mathbf{v}_5} f(5) | \\
\end{array} \]
Back to Ledrappier for $T^t$:

$$f(x, y) = 1 + x + y, \quad f^k(x, y) = 1 + x^k + y^k$$

$$t \in \mathbb{Z}_2 ^2 \iff t = \sum t_{k,e} x^k y^e$$

$$\rho_f(t) = t \cdot f^*$$

$$X_f = \ker \rho_f$$

Compute $n \times n$ periodic $p$ts using "external"

\[ (\xi, \eta) \in \mathbb{R}_n \times \mathbb{R}_n \rightarrow \tilde{v}_{\xi, \eta} \in \mathbb{C} \]

\[ (\tilde{v}_{\xi, \eta})_{k,e} = \xi^k \eta^e \]

\[ \tilde{p}_x (\tilde{v}_{\xi, \eta}) = \xi \cdot \tilde{v}_{\xi, \eta} \]

\[ \tilde{p}_y (\tilde{v}_{\xi, \eta}) = \eta \cdot \tilde{v}_{\xi, \eta} \]

\[ \tilde{p}_{1 + x + y} (\tilde{v}_{\xi, \eta}) = (1 + \xi + \eta) \tilde{v}_{\xi, \eta} = f(\xi, \eta) \tilde{v}_{\xi, \eta} \]

So

$$\det \tilde{p}_{1 + x + y} = \prod_{(\xi, \eta) \in \mathbb{R}_n^2} f(\xi, \eta)$$

$n^2$ eigenvalues
\[
\frac{1}{n^2} \log \# \text{ n x n per. pts} = \frac{1}{n^2} \sum_{(x, y) \in \mathbb{R}^2} \log |f(x, y)|
\]

\[
\rightarrow \int_{\mathbb{R}^2} \log |f| = m(f)
\]

2. There is a serious problem. Let \( z_3 = e^{2\pi i/3} \), so \( 1 + z_3 + z_3^2 = 0 \), and \( f(z_3, z_3^2) = 0 = f(z_3^2, z_3) \). So if \( 3 \ln \), two of the terms are \( \log 0 = -\infty \).

This is a recurring headache: various ways to deal with it.

- "Perturb" to something very similar, but which does not vanish
- Use the internal method and "integrate away" singularities
- Count periodic components instead of periodic points.
Graph of $\log |1 + e^{2\pi i s_1} + e^{2\pi i s_2}|$
Principal \( \mathbb{Z}^d \)-actions

\[
\mathbb{R}_d = \mathbb{Z} \left[ x_1, \ldots, x_d \right]
\]

\[
f(x_1, \ldots, x_d) \in \mathbb{R}_d
\]

\[
\prod_{i=1}^{d} x_i = k \alpha_f \quad \overset{\sim}{\leftrightarrow} \quad \mathbb{R}_d / \langle f \rangle
\]

\[
\alpha_f \quad \overset{\sim}{\leftrightarrow} \quad \langle x_1, x_2, \ldots, x_d \rangle
\]

**Thm** (L-S-W): \( h(\alpha_f) = m(f) = \log M(f) \).

**Lehmer's problem for \( \mathbb{R}_d \):**

\[
m(f(x, x^n)) \rightarrow m(f(x, y))
\]

so it's the same problem as for \( \mathbb{R}_1 \).

**Zero Entropy:** Generalize cyclotomic \( \Phi(x_1^{n_1} \cdots x_d^{n_d}) \) where \( \Phi(u) \) is cyclotomic

\[m(f) = 0 \iff f \text{ is a product of generalized cyclotomics}\]
Convergence of Riemann sums:

Let \( f(x, y) = 3 - x - x^{-1} - y - y^{-1} \)

\( f = 0 \) along a curve in \( \mathbb{S}^2 \), so \( \log |f| \)

has a curve of singularities.

The only roots of unity on \( C \) are \( (1, 5^t) \) and \( (5^t, 1) \).

Let \( F \subset \mathbb{S}^2 \) be a finite subgroup.

Does \( \frac{1}{|F|} \sum_{(5^n, n) \in F} \log |f(5^n, n)| \xrightarrow{?} m(f) \)

Very recent work shows this is true for "square" subgroups \( \mathbb{Z}_n \times \mathbb{Z}_n \).

There are two proofs, both very hard, one using logic and \( O \)-minimal sets!

The question for general \( F \)'s is still wide open.
Graph of \( \log |f| \) on \( S^2 \), where
\[
f(x, y) = 3 - x - x^{-1} - y - y^{-1}
\]
There is much more known about algebraic \( \mathbb{Z}^d \)-actions, and a fairly complete "dictionary", where dynamical properties can be translated into commutative algebra and algebraic geometry (and answered!).

But lest you think everything is known:

Consider \( M_2, M_3 : \mathbb{T} \to \mathbb{T} \), \( M_k(t) = kt \):

\[
\begin{align*}
\text{M}_2 & \quad \text{M}_3 \\
\begin{array}{c}
\begin{array}{c}
\color{red}{\ldots}
\end{array} \\
\end{array} & \\
\begin{array}{c}
\begin{array}{c}
\color{red}{\ldots}
\end{array} \\
\end{array} \\
\begin{array}{c}
\begin{array}{c}
\color{red}{\ldots}
\end{array} \\
\end{array} & \\
\begin{array}{c}
\begin{array}{c}
\color{red}{\ldots}
\end{array} \\
\end{array}
\end{align*}
\]

Haar measure is preserved by both. Some atomic measures too: \( \frac{1}{5} \sum_{j=0}^{5} \delta_{j/5} \). Are there any others?

Furstenberg's problem (or \( x_2, x_3 \))

Has been open for 50 years !!!