Lecture III: Algebraic actions of amenable gps

Generalize set-up from $\mathbb{Z}$, $\mathbb{Z}^d$ to general countable gp Γ, not necessarily abelian.

Γ: countable discrete group

$\mathbb{Z}\Gamma$: integral gp ring of Γ

$f \in \mathbb{Z}\Gamma$: $f = \sum_\gamma f_\gamma \cdot \gamma$ (finite sum, $f_\gamma \in \mathbb{Z}$), ring

$\mathbb{T}^\Gamma$: $t = \sum_\gamma t_\gamma \cdot \gamma$ (infinite sum, $t_\gamma \in \mathbb{T}$)

$\Gamma \curvearrowright \mathbb{T}^\Gamma$: $\theta \cdot t = \sum_\gamma t_\gamma \cdot \theta\gamma \quad (\theta \in \Gamma)$

$\widehat{\mathbb{T}^\Gamma} = \bigoplus_\gamma \mathbb{Z} = \mathbb{Z}\Gamma$

$f^* = \sum f_\gamma \cdot \gamma^{-1}$

$\rho_f(t) = t \cdot f^*$

left ideal gen by f

$X_f = \ker \rho_f \subset \mathbb{T}^\Gamma$, $\quad \widehat{X}_f = \mathbb{Z}\Gamma / \mathbb{Z}\Gamma \cdot f$

α_f : restriction of Γ-action to X_f

(X_f, α_f) : "Principal algebraic action of Γ"

<u>Examples</u>: If $\Gamma = \{x^n : n \in \mathbb{Z}\}$, Then $\mathbb{Z}\Gamma = \mathbb{Z}[x^{\pm}]$,

So $\mathbb{Z}\mathbb{Z} \cong \mathbb{Z}[x^{\pm}]$. Sim, $\mathbb{Z}\mathbb{Z}^d \cong \mathbb{Z}[x_1^{\pm}, \ldots, x_d^{\pm}]$

group

coefficients

<u>Main problem</u>: Compute $h(\alpha_f)$.

To even define (classical) h, need averaging sets.

A sequence $\{F_n\}$ of finite sets in Γ is <u>right-Følner</u>:

$$\frac{|F_n \triangle F_n \gamma|}{|F_n|} \to 0 \quad (n \to \infty)$$

Γ amenable: has a right Følner sequence.

<u>Simplest noncommutative Γ</u>:

$\mathbb{H}$: discrete Heisenberg gp

$$= \langle x, y, z \mid xz = zx,\ yz = zy,\ yx = xyz \rangle$$

$$\cong \left\{ \begin{bmatrix} 1 & b & c \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\}$$

corresponds to $x^a y^b z^c$

$$\mathbb{Z}\mathbb{H} = \left\{ \sum_{k,l,m} f_{k,l,m}\, x^k y^l z^m : f_{k,l,m} \in \mathbb{Z} \right\}$$

Basic information on algebraic $\mathbb{H}$-actions in a survey by L-Schmidt, available in course materials.

<u>Rk</u>: $\mathbb{H}$ is amenable, but "cubes" doen't work

Noncommutative groups are (much) harder than $\mathbb{Z}$, $\mathbb{Z}^d$:

- Left ideals in $\mathbb{Z}\Gamma$ are not 2-sided ideals
- $\mathbb{Z}\mathbb{H}$ is a domain ($fg=0 \Rightarrow f=0$ or $g=0$), but unique factorization into irreducibles can fail.
- $f \in \mathbb{Z}\mathbb{H}$ is not a function on a space, so no natural way to form $\int \log|f|$. We'll use f to construct an operator on $\ell^2(\Gamma)$ and use this operator to compute entropy.

2 <u>Zero divisor problem.</u> Spse $1 \neq \gamma \in \Gamma$ has $\gamma^n = 1$. Then in $\mathbb{Z}\Gamma$,

$$\underset{\neq 0}{(\gamma - 1)}\,\underset{\neq 0}{(\gamma^{n-1} + \gamma^{n-2} + \dots + \gamma + 1)} = \gamma^n - 1 = 0$$

¿ If $\gamma^n \neq 1$ for all $1 \neq \gamma \in \Gamma$ and all $n \geq 1$, can $\mathbb{Z}\Gamma$ have nonzero zero-divisors?

Incredibly, this problem is still open!

<u>Rk</u>: No zdv $\Leftrightarrow$ for $0 \neq f \in \mathbb{Z}\Gamma$ the map $g \mapsto fg$ is injective on $\mathbb{Z}\Gamma$, and you will see "injective" hypotheses in many theorems to avoid zdv problems.

<u>Some dynamical questions</u> when $\Gamma = \mathbb{H}$

- α_f ergodic? Ans: Always! (Ben Hayes, several proofs, one uses "ends" of groups)
- α_f mixing? No complete answer, several partial results:

  α_{1+x+y} is mixing

- α_f expansive? "Soft" answer general Γ:

$$t \in \ell^\infty(\Gamma) \longleftrightarrow t = \sum t_\gamma \cdot \gamma$$

$$t_\gamma \in \mathbb{R}, \ \sup_\gamma |t_\gamma| < \infty$$

$$t \in \ell^1(\Gamma): \ \sum_\gamma |t_\gamma| < \infty$$

$$\rho_f(t) = t \cdot f^* : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma)$$

<u>Soft Thm</u>: α_f expansive

$\Leftrightarrow$ ρ_f injective on $\ell^\infty(\Gamma)$

$\Leftrightarrow$ f invertible in $\ell^1(\Gamma)$

<u>Flavor of pf</u>: Spse α_f not expansive.

$\exists\, t \in X_f$ s.t. all t_γ very small.

$\neq 0$

$$\begin{tikzcd}
{[0,1)^\Gamma} \ni \tilde{t} \arrow[r, "\tilde{\rho}_f"] \arrow[d, "(\mathrm{mod}\ 1)"] & \text{all coord} \in \mathbb{Z} \arrow[d, "(\mathrm{mod}\ 1)"] \\
X_f \ni t \arrow[r, "\rho_f"] & 0
\end{tikzcd}$$

But $\tilde{\rho}_f(\tilde{t})$ has all coord. ① small, and ② integral $\Rightarrow \tilde{\rho}_f(\tilde{t}) = 0$, so ρ_f is not injective on $\ell^\infty(\Gamma)$.

Example: $f = 3 - x - y \in \mathbb{Z}\mathbb{H}$. Claim f is invertible in $\ell^1(\mathbb{H})$. Use geometric series!

$$\frac{1}{3-x-y} = \frac{1}{3}\,\frac{1}{1-(\frac{1}{3}x+\frac{1}{3}y)} = \frac{1}{3}\left\{1 + (\tfrac{1}{3}x+\tfrac{1}{3}y) + (\tfrac{1}{3}x+\tfrac{1}{3}y)^2 + \cdots\right\}$$

$$= \frac{1}{3} + \frac{1}{9}x + \frac{1}{9}y + \frac{1}{27}x^2 + \frac{1}{27}xy + \frac{1}{27}yx + \frac{1}{27}y^2 + \cdots$$

$$\in \ell^1(\mathbb{H})$$

What makes this work? f is "lopsided", i.e. $|f_{\gamma_0}| > \sum_{\gamma \neq \gamma_0} |f_\gamma|$

Sufficient but not necessary:

$\Gamma = \mathbb{Z}$, $f(x) = x^2 - x - 1$, not lopsided, but f is invertible in $\ell^1(\mathbb{Z})$.

Cool Theorem: If $f \in \mathbb{Z}\Gamma$ and α_f is expansive, then $\exists\, g \in \mathbb{Z}\Gamma$ s.t. fg is lopsided.

Open problem: Characterize those $f \in \mathbb{Z}H$ s.t. α_f is expansive. Is there a finite algorithm to decide this?

Historical Rk: $\Gamma = \mathbb{Z}$, $f \in \mathbb{Z}\mathbb{Z} = \mathbb{Z}[x^{\pm}]$.

Then α_f expansive iff $\hat{f}(e^{2\pi i s}) \neq 0$ all s.
Weiner's Thm $\Rightarrow \dfrac{1}{\hat{f}(e^{2\pi i s})}$ has absolutely convergent Fourier series, i.e. f is invertible in $\ell^1(\mathbb{Z})$. Proving this was the first triumph of commutative Banach algebras.

Entropy:

$$\boxed{h = \log \det}$$

Replace f as a function with f as an operator on $\ell^2(\Gamma)$, and use this operator to compute entropy.

Goal: Make operators and von Neumann algebras less terrifying !!

<u>Motivation</u>: Go back to $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ on $\mathbb{T}^2$,

or $f(x) = x^2 - x - 1 \in \mathbb{Z}\mathbb{Z}$, and (X_f, α_f).

To compute $p_n = \#$ n-per. pts, used f.d.

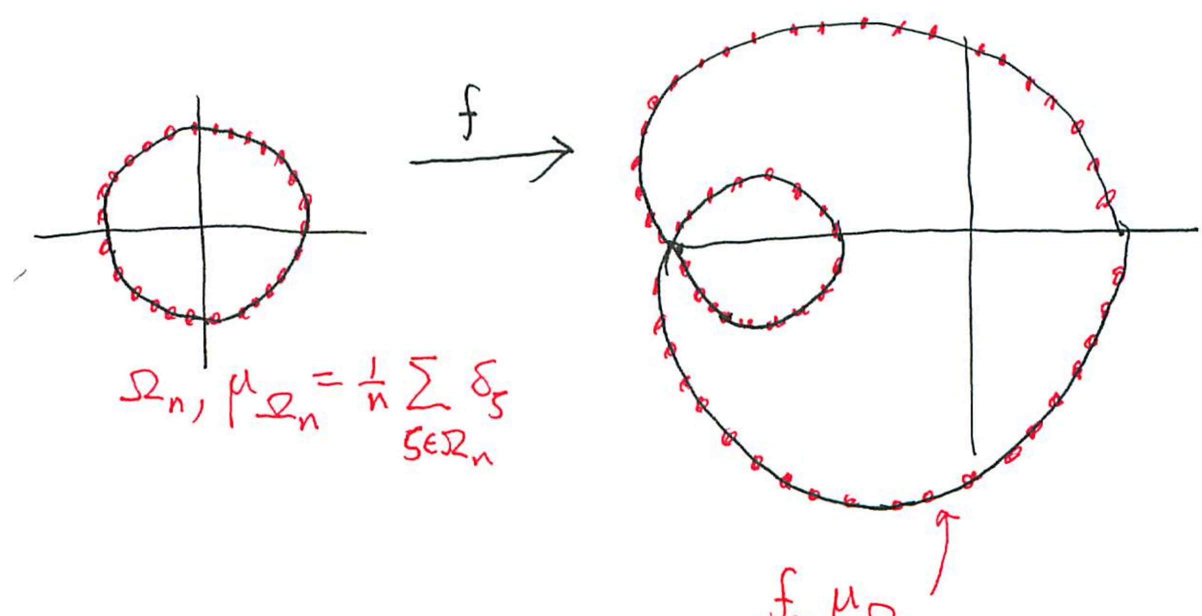
approx

$$\tilde{\rho}_f^{(n)} \text{ on } \mathbb{R}^{\mathbb{Z}/n\mathbb{Z}} : \begin{bmatrix} -1 & -1 & 1 & & & \\ & -1 & -1 & 1 & & \\ & & -1 & -1 & 1 & \\ & & & \ddots & \ddots & \ddots \\ & & & & & \\ -1 & 1 & & & & -1 \end{bmatrix}$$

circulant matrix

$$\tilde{\rho}_f^{(n)} \in \mathcal{B}(\ell^2(\mathbb{Z}/n\mathbb{Z}))$$

$$p_n = |\det \tilde{\rho}_f^{(n)}| = \Big|\prod_{\zeta \in \Omega_n} f(\zeta)\Big|$$



$$\frac{1}{n}\log P_n = \frac{1}{n}\log\left|\det \tilde{\rho}_f^{(n)}\right| = \frac{1}{n}\sum_{\zeta\in\Omega_n}\log|f(\zeta)|$$

$$= \int_{\mathbb{C}} \log|z|\, d(f_* \mu_{\Omega_n})$$

Now let "$n\to\infty$":

$$\ell^2(\mathbb{Z}/n\mathbb{Z}) \text{ ``}\longrightarrow\text{'' } \ell^2(\mathbb{Z})$$

$$\tilde{\rho}_f^{(n)} \longrightarrow \rho_f$$

$$\Omega_n \longrightarrow \mathbb{S}$$

$$\mu_{\Omega_n} \longrightarrow \mu_{\mathbb{S}} \quad \text{Haar meas}$$

$$f_*\mu_{\Omega_n} \longrightarrow f_*\mu_{\mathbb{S}}$$

$$(= \mu_f \text{ on } f(\mathbb{S}))$$

$$\ell^2(\mathbb{Z}) \overset{\wedge}{\longleftrightarrow} L^2(\mathbb{T})$$

$\rho_f \in \mathcal{B}(\ell^2(\mathbb{Z})) \longleftrightarrow$ mult. by $\hat{f}(e^{2\pi i s})$ on $L^2(\mathbb{T})$

i.e. $T_f \in \mathcal{B}(L^2(\mathbb{T}))$

$$(T_f(\phi))(s) = f(e^{2\pi i s})\,\phi(s)$$

T_f is a normal operator (commutes with T_f^*)

spectrum $T_f = \{ z \in \mathbb{C} : T_f - zI \text{ not invertible} \}$

$= f(\mathbb{S})$

spectral measure: $f_* \mu_{\mathbb{S}} =: \mu_f$ on $f(\mathbb{S})$

Using duality, we've recast computing $h(\alpha_f)$ into properties of the operator $\rho_f \in \mathcal{B}(\ell^2(\mathbb{Z}))$:

$$h(\alpha_f) = \int_{\mathbb{C}} \log|z| \, d\mu_f(z) \quad \left(= \int_{\mathbb{S}} \log|f(s)| \, ds \right)$$

How to define the det of an operator?

$$\begin{tikzcd} \left(\det \tilde{\rho}_f^{(n)}\right)^{\frac{1}{n}} \arrow[r] \arrow[d] & \exp\left[\int_{\mathbb{C}} \log|z| \, d\mu_f(z) \right] \arrow[dl, dashed] \\ \text{"} \det \rho_f \text{"} & \end{tikzcd}$$

Use the spectral measure of T_f (or ρ_f) to define det ρ_f.

This is exactly how Fuglede and Kadison defined det for certain classes of operators.

The general setting: $T_n \xrightarrow{SOT} T \Leftrightarrow T_n h \to Th$ all h

$\mathcal{L}\Gamma$: strong operator top closure of $\{\rho_f : f \in \mathbb{C}\Gamma\}$
von Neumann alg in $\mathcal{B}(\ell^2(\Gamma))$.

<u>Ex</u>: $\Gamma = \mathbb{Z}$

$$\ell^2(\mathbb{Z}) \xleftrightarrow{\ \wedge\ } L^2(\mathbb{T})$$

$$\rho_f \qquad\qquad T_f$$

$$\rho_{f_n} \xrightarrow{SOT} \rho_f \qquad\qquad \hat{f}_n(e^{2\pi i s})\phi(t) \xrightarrow{L^2} \hat{f}(e^{2\pi i s})\phi(t)$$

all $\phi \in L^2(\mathbb{T})$

$$\mathcal{L}\mathbb{Z} \qquad\qquad L^\infty(\mathbb{T})$$

So here $\mathcal{L}\mathbb{Z}$ consists of all convolution operators defined by Fourier series of L^∞-fcns

<u>Rk</u>: If we had used the norm top instead of SOT, would get the uniform closure of polynomials, i.e. $C(\mathbb{T})$ instead of $L^\infty(\mathbb{T})$. This is called $C^*(\mathbb{T})$.

<u>Trace tr on $\mathcal{L}\Gamma$</u>: $\operatorname{tr} T = \langle T\delta_e, \delta_e \rangle$
e = identity of Γ. For $f \in \mathbb{Z}\Gamma$,
$\operatorname{tr} \rho_f = f_e$ = "constant term" of f.

$$\operatorname{tr}(ST) = \operatorname{tr}(TS).$$

Start with $T \in \mathcal{L}\Gamma$. Form T^*T, self-adj.

roughly: If $T = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{bmatrix}$, $T^*T = \begin{bmatrix} |\lambda_1|^2 & & \\ & \ddots & \\ & & |\lambda_r|^2 \end{bmatrix}$

$\mathrm{spec}(T^*T) \subset [0, \|T\|^2] \subset [0, \infty)$

cts functional calculus $\rightsquigarrow$

$$|T| = \sqrt{T^*T}$$

spectral measure $\mu_{|T|}$ on $[0, \|T\|]$

$$\det_{\mathcal{L}\Gamma} T := \exp\left(\mathrm{tr} \log |T|\right)$$
$$= \exp\left(\int_0^{\|T\|} \log t \, d\mu_{|T|}(t)\right)$$

For $T = \rho_f$,

$$\log \det_{\mathcal{L}\Gamma} \rho_f = \int \log t \, d\mu_{|\rho_f|}(t)$$
$$= \mathrm{tr} \log |f|$$

<u>Thm</u>: Let Γ be a countable amenable gp, $f \in \mathbb{Z}\Gamma$, and identify f with the operator $\rho_f \in \mathcal{B}(\ell^2(\Gamma))$. Then

$$h(\alpha_f) = \log \det_{\mathcal{L}\Gamma} f.$$

¿ Characterize $f \in \mathbb{Z}\Gamma$ with $h(\alpha_f)=0$?
Is There an algorithm (even for $\Gamma = \mathbb{H}$)?

¿ Numerically compute $h(\alpha_f)$?

Rk: For a long time it was not known whether $h(\alpha_f) = h(\alpha_{f^*})$! This is true, but hard.

For some f's, $h(\alpha_f)$ can be calculated.

① Lopsided, non recurrent.

$\Gamma = \mathbb{H}$, $f = 3 - x - y$. Compute tr log f

$$\log(1-u) = -\left(u + \frac{u^2}{2} + \frac{u^3}{3} + \cdots\right)$$

$$\log f = \log\left[3\left(1 - \tfrac{1}{3}(x+y)\right)\right]$$

$$= \log 3 - \tfrac{1}{3}(x+y) - \tfrac{1}{2}\left[\tfrac{1}{3}(x+y)\right]^2 - \cdots$$

$$\operatorname{tr} \log f = \log 3 = h(\alpha_f).$$

② Lopsided, recurrent: Ten Martini Problem

$\Gamma = \mathbb{H}$, $f = 5 - x - x^{-1} - y - y^{-1}$

$$= 5[1 - g] \qquad g = \tfrac{1}{5}(x + x^{-1} + y + y^{-1})$$

$$\log f = \log 5 - \sum_{n=1}^{\infty} \frac{g^n}{n}$$

These can contribute to tr

The contributions of $\frac{g^n}{n}$ to tr depend on the "random walk" on Γ.

$$\Gamma = \mathbb{Z}^2 : \quad 1.5079 = m(f)$$

$$\Gamma = \mathbb{H} : \quad 1.514708$$

$$\Gamma = \mathbb{F}_2 : \quad 1.514787$$
$$= \log\left[\tfrac{1}{18}(35 + 13\sqrt{13})\right]$$

Ex: For $0 \leq s, t < 1$ put

$$F(s,t) := \limsup_{n\to\infty} \frac{1}{n} \log \left\| \prod_{k=0}^{n-1} \begin{bmatrix} 0 & 1 \\ 1 & e^{2\pi i(s+kt)} \end{bmatrix} \right\|$$

$$F(0,0) = \log \frac{1+\sqrt{5}}{2}$$

Using $f = y^2 - xy - 1 \in \mathbb{Z}\mathbb{H}$, can deduce:

- $h(\alpha_f) = 0$
- $F(s,t) \geq 0 \quad (\det = 1)$
- $h(\alpha_f) = \int_0^1 \int_0^1 F(s,t)\, ds\, dt.$

$\therefore F(s,t) = 0$ a.e. $(s,t) \in [0,1)^2$!

This seems to be new, even to experts.

Open: Replace $s+kt$ with $s+k^2t$, numerically $F(s,t) > 0$, is this true?