Lecture IV: Algebraic actions of sofic gps

The algebraic set-up is exactly the same:

$$\Gamma,\ \mathbb{Z}\Gamma,\ \mathbb{Z}\Gamma/\mathbb{Z}\Gamma\cdot f,\ X_f,\ \alpha_f .$$

But even to define entropy if Γ is not amenable requires new ideas.

If Γ is sofic, there are sofic approximations $\Sigma = \{\sigma_i : \Gamma \to \mathrm{sym}(d_i)\}$ and can use these to define sofic entropy for Γ-actions, in particular for α_f ($f \in \mathbb{Z}\Gamma$).

If we identify f with the convolution operator ρ_f on $\ell^2(\Gamma)$, then the F-K det $\det_{\mathcal{L}\Gamma} f$ is defined

Main Theorem (Hayes):

$$h_\Sigma(\alpha_f) = \log \det_{\mathcal{L}\Gamma} f .$$

In particular, all Σ give same sofic entropy.

Very nice presentation of the proof of this in [Kerr-Li; Chap 14].

Roughly: $h_\Sigma(\alpha_f)$ is the growth rate of certain finite sets, and these sets can also be used to compute $\det_{\mathcal{L}\Gamma} f$.

One slick technical trick is to perturb away finite-dimensional approximations having 0 eigenvalues.

<u>But</u>, computing, or even approximating, $\det_{\mathcal{L}\Gamma} f$ is in general very hard.

<u>Focus on the case $\Gamma = \mathbb{F}_2 = \langle x, y \rangle$ free gp.</u>

<u>Rk</u>: $\mathbb{F}_2$ is <u>residually finite</u>:

$\exists$ finite-index normal subgps $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \cdots$ with $\bigcap_{n=1}^{\infty} \Gamma_n = \{e\}$.

Then use $\sigma_i : \mathbb{F}_2 \to \mathbb{F}_2/\Gamma_i$, where each elt of $\mathbb{F}_2/\Gamma_i$ thought of as a permutation in $\mathrm{sym}(\mathbb{F}_2/\Gamma_i)$.

Here's a more "random" approach:

Let N be large, and choose two "random" permutation matrices $P_N, Q_N \in \mathrm{sym}(N)$. Then $x \mapsto P_N$, $y \mapsto Q_N$ define a map $\mathbb{F}_2 \longrightarrow \mathrm{sym}(N)$ (since $\mathbb{F}_2$ is free).

First surprise: P_N and Q_N will generate* all of $\mathrm{sym}(N)$ with probability $\to 1$ as $N \to \infty$

* Except if P_N and Q_N are both even, then they generate A_N with high probability.

So for $f \in \mathbb{Z}\mathbb{F}_2$, should be able to get finite approximations to $\mu_{|f|}$:

$$f^*(P_N, Q_N)\, f(P_N, Q_N) \qquad N \times N \text{ matrix}$$

Eigenvalues $\lambda_1, \dots, \lambda_N \in [0, \|f\|^2]$

$$\frac{1}{N} \sum_{\substack{j=1 \\ |\lambda_j| \neq 0}}^{N} \log |\lambda_j| \longrightarrow \int_0^{\|f\|^2} \log t \, d\mu_{|f|^2}(t)$$

$$= \log \det |f|^2 = 2 \log \det |f|$$

$$= 2 h_\Sigma(\alpha_f)$$

Do this for $1+x+y$:

$$(I+P_N^*+Q_N^*)\cdot(I+P_N+Q_N)$$

Numerically, $\frac{1}{N}\sum \log|\lambda_j| \approx 0.28$

indicating $h(\alpha_{1+x+y}) \approx 0.14$

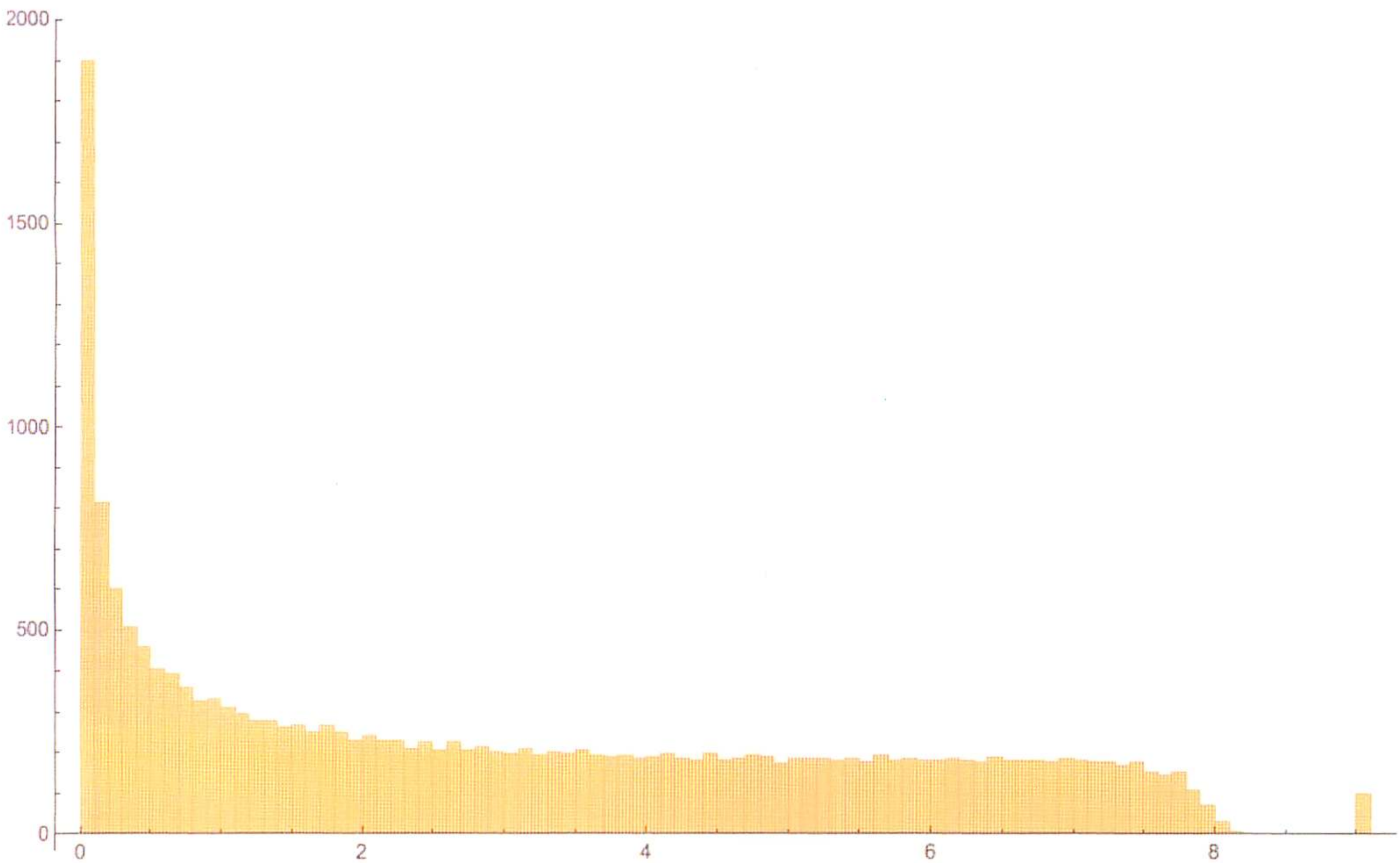
But why should the distribution μ_N of eigenvalues $\xrightarrow{w^*} \mu_{|f|^2}$?

<u>Moment method</u>: Spse μ_n, μ measures on $[0,A]$. Define the $(2m)^{th}$ moment $M_{2m}(\mu) := \int_0^A t^{2m}\,d\mu(t)$.

Then $\mu_N \xrightarrow{w^*} \mu_{|f|^2}$ iff for each m

$$M_{2m}(\mu_N) \to M_{2m}(\mu_{|f|^2}).$$

[This is just SW: polynomials in even powers of t are dense in $C[0,A]$.]



Distribution of eigenvalues of

$$(I + P_N^* + Q_N^*)(I + P_N + Q_N)$$

[N=200, repeated 100 times]

The $(2m)^{Th}$ moment of f^*f is easy to compute:

$$\mathrm{tr}\,(f^*f)^m = \text{constant term of } (f^*f)^m,$$

a finite calculation.

What about $f_N = f(P_N, Q_N)$?

$$\mathrm{tr}(f_N^* f_N)^m \geq \mathrm{tr}(f^*f)^m$$

every contribution to tr occurs here

could have a few "accidental" contributions to tr due to finite approximations

<u>Ex</u>: $f(x,y) = 1 + x + y$, $m = 1$ $\quad \mathrm{tr}(f^*f) = 3$

$N = 20$: 3.6, 3.1, 3.4, 3.3, 3.6, 3.3

$N = 500$: 3.004 3.008 3.024, 3.008, ...

Is There a "slick" formula for h/α_{1+x+y}?

Yes! Uses free probability Theory, noncommutative independence.
This computes moments, and identifies The corresponding measure.

Thm: For $1+x+y \in \mathbb{Z}\mathbb{F}_2$,

$$h(\alpha_{1+x+y}) = \frac{1}{2}\log\left(\frac{4}{3}\right) \cong 0.143841$$

In fact, for $1+x_1+\cdots+x_r \in \mathbb{Z}\mathbb{F}_r$

r	$h(\alpha_{1+x_1+\cdots+x_r})$
2	$\log\left(\frac{2}{\sqrt{3}}\right)$
3	$\log\left(\frac{3\sqrt{3}}{4}\right)$
4	$\log\left(\frac{16}{5\sqrt{5}}\right)$
$\vdots$	$\vdots$
29	$\frac{19\log 19}{2} - 9\log 20$

If $$\phi(d) := \int_0^{\sqrt{d}} \frac{2d^2(d-1)\, r\log r}{(d^2-r^2)^2}\,dr$$

then $h(\alpha_{1+x_1+\cdots+x_r}) = \phi(r+1)$

<u>Lopsided example:</u>

$$f = 5 - x - x^{-1} - y - y^{-1} \in \mathbb{Z}\mathbb{F}_2$$

We saw in Lecture III we can compute $\log f$ using the power series for $\log(1-u)$:

$$\log f = \log 5 + \log(1 - g) \qquad g = \tfrac{1}{5}(x + x^{-1} + y + y^{-1})$$

$$= \log 5 - \sum_{n=1}^{\infty} \frac{g^n}{n}$$

In $\mathbb{F}_2$ there are explicit counts of the number of words in x, x^{-1}, y, y^{-1} of length n whose product is e.

$$h(\alpha_f) = \log \det_{\mathcal{L}\mathbb{F}_2} f$$

$$= \operatorname{tr} \log f = \log \left[\tfrac{1}{18}(35 + 13\sqrt{13}) \right]$$

Another lopsided example:

$$f = 3 - x - y \in \mathbb{Z}\mathbb{F}_2$$

$$\log f = \log 3 - \sum_{n=1}^{\infty} \frac{(\frac{1}{3}x + \frac{1}{3}y)^n}{n}$$

$$h(\alpha_f) = \operatorname{tr} \log f = \log 3.$$

¿ Is α_f measurably isomorphic to the Bernoulli 3-shift over $\mathbb{F}_2$?

There is a candidate isomorphism using homoclinic points¡

We've only started to scratch the surface of algebraic actions of sofic groups, and there are many, many open problems!!!