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Dedicated to Anatoly Vershik on the occasion of his 80th birthday

A survey of algebraic actions of the discrete Heisenberg group

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Abstract. The study of actions of countable groups by automorphisms of compact Abelian groups has recently undergone intensive development, revealing deep connections with operator algebras and other areas. The discrete Heisenberg group is the simplest non-commutative example, where dynamical phenomena related to its non-commutativity already illustrate many of these connections. The explicit structure of this group means that these phenomena have concrete descriptions, which are not only instances of the general theory but are also testing grounds for further work. This paper surveys what is known about such actions of the discrete Heisenberg group, providing numerous examples and emphasizing many of the open problems that remain.

Bibliography: 71 titles.

Keywords: algebraic action, Heisenberg group, expansiveness, entropy.

Contents

1. Introduction

Since Halmos's observation [29] in 1943 that automorphisms of compact groups automatically preserve Haar measure, these maps have provided a rich class of examples in dynamics. In case the group is Abelian, its dual group is a module over

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the Laurent polynomial ring $\mathbb{Z}[x^{\pm}]$. Such modules have a well-developed structure theory, which enables a comprehensive analysis of general automorphisms in terms of basic 'building blocks' that can be completely understood.[1]

The roots of the study of several commuting algebraic maps can be traced back to the seminal 1967 paper of Furstenberg [22], where he considered the joint dynamical properties of multiplication by different integers on the additive torus. In 1978 Ledrappier [37] gave a simple example of a mixing action of $\mathbb{Z}^2$ by automorphisms of a compact Abelian group that was not mixing of higher orders. For an action by d commuting automorphisms, the dual group is a module over the Laurent polynomial ring $\mathbb{Z}[u_1^{\pm}, \dots, u_d^{\pm}]$, that is, the integral group ring $\mathbb{Z}\mathbb{Z}^d$ of $\mathbb{Z}^d$. The commutative algebra of such modules provides effective machinery for analyzing such actions. This point of view was initiated in 1989 by Kitchens and the second author [36], and a fairly complete theory of the dynamical properties of such actions is now available [60].

Let Δ denote an arbitrary countable group, and let α denote an action of Δ by automorphisms of a compact Abelian group, or an *algebraic* Δ*-action*. The initial steps in analyzing such actions were taken in [60], Chap. 1 and give general criteria for some basic dynamical properties such as ergodicity and mixing.

In 2001 Einsiedler and Rindler [18] studied the special case when $\Delta = \Gamma$, the discrete Heisenberg group, as a first step towards algebraic actions of non-commutative groups. Here the concrete nature of Γ suggests that there should be specific answers to the natural dynamical questions, and they give several instances of this together with instructive examples. However, the algebraic complexity of the integral group ring $\mathbb{Z}\Gamma$ prevents the comprehensive analysis available in the commutative case.

A dramatic new development occurred in 2006 with the work of Deninger on entropy for principal Δ-actions. Let $f \in \mathbb{Z}\Delta$, and let $\mathbb{Z}\Delta f$ denote the principal left ideal generated by f. Then Δ acts on the quotient $\mathbb{Z}\Delta/\mathbb{Z}\Delta f$, and there is a dual Δ-action α_f on the compact dual group, called a *principal* Δ*-action*. Deninger showed in [15] that in many cases the entropy of α_f equals the logarithm of the Fuglede–Kadison determinant of the linear operator corresponding to f on the group von Neumann algebra of Δ. In case $\Delta = \mathbb{Z}^d$, this reduces to the calculation in [44] of entropy in terms of the logarithmic Mahler measure of f. Subsequent work by Deninger, Li, Schmidt, Thom, and others shows that this and related results hold in great generality (see, for example, [17], [38], and [40]). In [40] the authors proved that three different concepts connected with Δ-actions, namely, entropy, Fuglede–Kadison determinants, and L^2-torsion, coincide, revealing deep connections that are only partly understood.

[1] *Russian editor's note*: It clearly was not the authors' intention to give a survey of the whole of algebraic dynamics, which is a significant chapter of both ergodic theory and the theory of groups of algebraic automorphisms. Their goal is more specific. Nevertheless, in connection with the pioneering works of Halmos one cannot fail to mention that at about the same time several papers of V. A. Rohlin on ergodic theory appeared, containing many important results on the ergodic and spectral properties of automorphisms of compact Abelian groups. Subsequently, Rohlin and a whole group of his followers proved fundamental results on the entropy and other properties of such automorphisms.

These ideas have some interesting consequences. For example, by computing the entropy of a particular Heisenberg action in two different ways, we can show that

$$\lim_{n\to\infty} \frac{1}{n} \log \left\| \prod_{k=0}^{n-1} \begin{bmatrix} 0 & 1 \\ 1 & e^{2\pi i(ka+b)} \end{bmatrix} \right\| = 0 \tag{1.1}$$

for almost every pair (a, b) of real numbers. Despite its simplicity, this fact does not appear to follow from known results on random matrix products.

Our purpose here is to survey what is known for the Heisenberg case $\Delta = \Gamma$, and to point out many of the remaining open questions. As Γ is the simplest non-commutative example (other than finite extensions of $\mathbb{Z}^d$, which are too close to the Abelian case to be interesting), any results will indicate limitations of what a general theory can accomplish. Also, the special structure of Γ should enable explicit answers to many questions, and yield particular examples of various dynamical phenomena. It is also quite instructive to see how a very general machinery, used for algebraic actions of arbitrary countable groups, can be made quite concrete for the case of Γ. We hope to inspire further work by making this special case both accessible and attractive.

2. Algebraic actions

Let Δ be a countable discrete group. The *integral group ring* $\mathbb{Z}\Delta$ of Δ consists of all finite sums of the form $g = \sum_\delta g_\delta \delta$ with $g_\delta \in \mathbb{Z}$, equipped with the obvious ring operations inherited from the multiplication in Δ. The *support* of g is the subset $\mathrm{supp}(g) = \{\delta \in \Delta : g_\delta \neq 0\}$.

Suppose that Δ acts by automorphisms of a compact Abelian group X. Such actions are called *algebraic Δ-actions*. Denote the action of $\delta \in \Delta$ on $t \in X$ by $\delta \cdot t$. Let M be the (discrete) dual group of X, with additive dual pairing denoted by $\langle t, m\rangle$ for $t \in X$ and $m \in M$. Then M becomes a module over $\mathbb{Z}\Delta$ by defining $\delta \cdot m$ to be the unique element of M so that $\langle t, \delta \cdot m\rangle = \langle \delta^{-1} \cdot t, m\rangle$ for all $t \in X$, and extending this to additive integral combinations in $\mathbb{Z}\Delta$.

Conversely, if M is a $\mathbb{Z}\Delta$-module, its compact dual group $X_M = \widehat{M}$ carries a Δ-action α_M dual to the Δ-action on M. Thus, there is a 1-1 correspondence between algebraic Δ-actions and $\mathbb{Z}\Delta$-modules.

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the additive torus. Then the dual group of $M = \mathbb{Z}\Delta$ can be identified with $\mathbb{T}^\Delta$ via the pairing $\langle t, g\rangle = \sum_\delta t_\delta g_\delta$ where $t = (t_\delta) \in \mathbb{T}^\Delta$ and $g = \sum_\delta g_\delta \delta \in \mathbb{Z}\Delta$.

For $\theta \in \Delta$, the action of θ on a $t \in \mathbb{T}^\Delta$ is defined via duality by $\langle \theta \cdot t, g\rangle = \langle t, \theta^{-1} \cdot g\rangle$ for all $g \in \mathbb{Z}\Delta$. By taking $g = \delta \in \Delta$, we get that $(\theta \cdot t)_\delta = t_{\theta^{-1}\delta}$. It is sometimes convenient to think of elements in $\mathbb{T}^\Delta$ as infinite formal sums $t = \sum_\delta t_\delta \delta$, and then $\theta \cdot t = \sum_\delta t_\delta \theta\delta = \sum_\delta t_{\theta^{-1}\delta}\delta$. This allows a well-defined multiplication of elements in $\mathbb{T}^\Delta$ by elements from $\mathbb{Z}\Delta$, both on the left and on the right.

We remark that the shift-action $(\theta \cdot t)_\delta = t_{\theta^{-1}\delta}$ is opposite to the traditional shift direction when Δ is $\mathbb{Z}$ or $\mathbb{Z}^d$, but is forced when Δ is non-commutative. This has sometimes caused confusion; for example, the last displayed equation in [18], p. 118 is not correct.

Now fix $f \in \mathbb{Z}\Delta$. Let $\mathbb{Z}\Delta f$ be the principal left ideal generated by f. The quotient module $\mathbb{Z}\Delta/\mathbb{Z}\Delta f$ has dual group $X_f \subset \mathbb{T}^\Delta$. An element $t \in \mathbb{T}^\Delta$ is in X_f if and only if $\langle t, gf\rangle = 0$ for all $g \in \mathbb{Z}\Delta$. This is equivalent to the condition that $\langle t \cdot f^*, g\rangle = 0$ for all $g \in \mathbb{Z}\Delta$, where $f^* = \sum_\delta f_\delta \delta^{-1}$. Hence, $t \in X_f$ exactly when $t \cdot f^* = 0$, with the above conventions for right multiplication of elements in $\mathbb{T}^\Delta$ by members of $\mathbb{Z}\Delta$. In other words, if we define $\rho_f(t) = t \cdot f^*$ to be right convolution by f^*, then X_f is the kernel of ρ_f. In terms of coordinates, t is in X_f precisely when $\sum_\delta t_{\theta\delta} f_\delta = 0$ for all $\theta \in \Delta$.

Our focus here is on the discrete Heisenberg group Γ, generated by elements x, y, and z subject to the relations $xz = zx$, $yz = zy$, and $yx = xyz$. Alternatively, Γ is the subgroup of $SL(3,\mathbb{Z})$ generated by the elements

$$x \leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad y \leftrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad z \leftrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We will sometimes use the notation R for the integral group ring $\mathbb{Z}\Gamma$ of Γ when emphasizing its ring-theoretic properties. The center of Γ is $\mathrm{Z} = \{z^k : k \in \mathbb{Z}\}$. The center of R is then the Laurent polynomial ring $\mathbb{Z}\mathrm{Z} = \mathbb{Z}[z^{\pm}]$. Hence, every element of R can be written as

$$g = \sum_{k,l,m} g_{klm} x^k y^l z^m = \sum_{k,l} g_{kl}(z) x^k y^l,$$

where $g_{klm} \in \mathbb{Z}$ and $g_{kl}(z) \in \mathbb{Z}\mathrm{Z}$. For $g = \sum_{k,l} g_{kl}(z) x^k y^l \in \mathbb{Z}\Gamma$, define the *Newton polygon* $\mathscr{N}(g)$ of g to be the convex hull in $\mathbb{R}^2$ of those points (k,l) for which $g_{kl}(z) \neq 0$. In particular, $\mathscr{N}(0) = \varnothing$. Because $\mathbb{Z}\mathrm{Z}$ is an integral domain, it is easy to verify that $\mathscr{N}(gh)$ equals the Minkowski sum $\mathscr{N}(g) + \mathscr{N}(h)$ for all $g, h \in \mathbb{Z}\Gamma$. This shows that $\mathbb{Z}\Gamma$ has no non-trivial zero-divisors. However, a major difference between the commutative case and $\mathbb{Z}\Gamma$ is that unique factorization into irreducibles fails for $\mathbb{Z}\Gamma$.

Example 2.1. It is easy to verify that

$$(y-1)(y-z)(x+1) = (xyz^2 - xz + y - z)(y-1).$$

Each of the linear factors is clearly irreducible by a Newton polygon argument. We claim that $f(x,y,z) = xyz^2 - xz + y - z$ cannot be factored in $\mathbb{Z}\Gamma$. Note that $\mathscr{N}(f) = [0,1]^2$. Suppose that $f = gh$. Adjusting by units and reordering factors if necessary, we may assume that g and h have the form $g(x,y,z) = g_0(z) + g_1(z)x$ and $h(x,y,z) = h_0(z) + h_1(z)y$. Expanding gh, we find that

$$g_0(z)h_0(z) = -z, \quad g_0(z)h_1(z) = 1, \quad g_1(z)h_0(z) = -z, \quad g_1(z)h_1(z) = z^2.$$

Hence, $g_0(z) = h_1(z)^{-1} = g_1(z)z^{-2}$ and $h_0(z) = -zg_1(z)^{-1}$. Then we would have

$$-z = g_0(z)h_0(z) = \big(g_1(z)z^{-2}\big)\big(-zg_1(z)^{-1}\big) = -z^{-1},$$

which is not true. This proves that f has no non-trivial factorizations in $\mathbb{Z}\Gamma$.

Since Γ is nilpotent of rank 2, it is polycyclic, and so R is both left- and right-Noetherian, that is, R satisfies the ascending chain condition on both left ideals and on right ideals [56].

Suppose now that M is a finitely generated left R-module, say generated by $m_1, \dots, m_l$. The map $R^l \to M$ defined by $[g_1, \dots, g_l] \mapsto g_1 m_1 + \cdots + g_l m_l$ is surjective. Its kernel K is a left R-submodule of the Noetherian module R^l, hence also finitely generated, say by $[f_{11}, \dots, f_{1l}], \dots, [f_{k1}, \dots, f_{kl}]$. Let $F = [f_{ij}] \in R^{k\times l}$ be the rectangular matrix whose rows are the generators of K. Then $K = R^k F$, and $M \cong R^l/R^k F$. We will denote the corresponding algebraic Γ-action for this presentation of M by α_F. Note that when $k = l = 1$ we are reduced to the case $F = [f]$, corresponding to the quotient module R/Rf and the principal Γ-action α_f.

3. Ergodicity

Let X be a compact Abelian group and let μ denote Haar measure on X, normalized so that $\mu(X) = 1$. If ϕ is a continuous automorphism of X, then the measure ν defined by $\nu(E) = \mu\big(\phi(E)\big)$ is also a normalized translation-invariant measure. Hence $\nu = \mu$, and μ is ϕ-invariant.

This shows that if α is an algebraic action of a countable group Δ on X, then α is μ-measure-preserving. A measurable set $E \subset X$ is said to be *α-invariant* if $\alpha^\delta(E)$ agrees with E off a null set for every $\delta \in \Delta$. The action α is *ergodic* if the only α-invariant sets have measure 0 or 1. The following, which is a special case of a result due to Kaplansky [33], gives an algebraic characterization of ergodicity.

Lemma 3.1 ([60], Lemma 1.2). *Let Δ be a countable discrete group, and α be an algebraic Δ-action on a compact Abelian group X whose dual group is M. Then α is ergodic if and only if the Δ-orbit of every non-zero element of M is infinite.*

Roughly speaking, this result follows from the observation that the existence of a bounded measurable α-invariant function on X is equivalent to the existence of a non-zero finite Δ-orbit in M.

For actions of the Heisenberg group Γ, this raises the question of characterizing those $F \in R^{k\times l}$ for which α_F is ergodic. The first result in this direction is due to Hayes.

Theorem 3.2 ([30], Theorem 2.3.6). *For every $f \in \mathbb{Z}\Gamma$ the principal algebraic Γ-action α_f is ergodic.*

Proof. We give a brief sketch of the proof. First recall that $\mathbb{Z}Z$ is a unique factorization domain. Define the *content* $\mathsf{c}(g)$ of an element $g = \sum_{i,j} g_{ij}(z)x^i y^j \in \mathbb{Z}\Gamma \setminus \{0\}$ to be the greatest common divisor in $\mathbb{Z}Z$ of the non-zero coefficient polynomials $g_{ij}(z)$, and put $\mathsf{c}(0) = 0$. A simple variant of the proof of Gauss's lemma shows that $\mathsf{c}(gh) = \mathsf{c}(g)\mathsf{c}(h)$ for all $g, h \in \mathbb{Z}\Gamma$.

Now fix $f \in \mathbb{Z}\Gamma = R$. The case $f = 0$ is trivial, so assume that $f \neq 0$. Suppose that $h + Rf$ has finite Γ-orbit in R/Rf. Then there are $m, n \geqslant 1$ such that $(x^m - 1)h = g_1 f$ and $(z^n - 1)h = g_2 f$ for some $g_1, g_2 \in R$. And then

$$\mathsf{c}\big((x^m - 1)h\big) = \mathsf{c}(x^m - 1)\mathsf{c}(h) = \mathsf{c}(h) = \mathsf{c}(g_1)\mathsf{c}(f),$$

so that $\mathsf{c}(f)$ divides $\mathsf{c}(h)$ in $\mathbb{Z}\mathrm{Z}$. Also,

$$\mathsf{c}\big((z^n-1)h\big) = (z^n-1)\mathsf{c}(h) = \mathsf{c}(g_2)\mathsf{c}(f),$$

so that $(z^n-1)\big[\mathsf{c}(h)/\mathsf{c}(f)\big] = \mathsf{c}(g_2)$, and hence z^n-1 divides $\mathsf{c}(g_2)$. Therefore, $g_2/(z^n-1) \in R$, and so $h = \big[g_2/(z^n-1)\big]f \in Rf$, showing that $h + Rf = 0$ in R/Rf. □

Hayes called a group Δ *principally ergodic* if every principal algebraic Δ-action is ergodic. He extended Theorem 3.2 to show that the following classes of groups are principally ergodic: torsion-free nilpotent groups that are not virtually cyclic (that is, do not contain a cyclic subgroup of finite index), free groups on more than one generator, and groups that are not finitely generated. Clearly, $\mathbb{Z}$ is not principally ergodic, since for example the action of x on the module $\mathbb{Z}[x^{\pm}]/\langle x^k-1\rangle$ dualizes to a $k \times k$ permutation matrix on $\mathbb{T}^k$, which is not ergodic.

Recently Li, Peterson, and the second author used a very different approach to proving principal ergodicity, based on cohomology [39]. These methods greatly increased the collection of countable discrete groups known to be principally ergodic, including all such groups that contain a finitely generated amenable subgroup that is not virtually cyclic.

We will now describe how their ideas work in the case of Γ. We begin by describing two important properties of finite-index subgroups of Γ, namely, that they are amenable and have only one end.

For an arbitrary discrete group Δ let

$$\ell^1(\Delta,\mathbb{R}) = \left\{ w \in \mathbb{R}^\Delta \colon \|w\|_1 := \sum_\delta |w_\delta| < \infty \right\},$$

$$\ell^\infty(\Delta,\mathbb{R}) = \left\{ w \in \mathbb{R}^\Delta \colon \|w\|_\infty := \sup_\delta |w_\delta| < \infty \right\},$$

so that $\ell^\infty(\Delta,\mathbb{R})$ is the dual space to $\ell^1(\Delta,\mathbb{R})$.

Fix $K, L \geqslant 1$, and put $M = KL$. Let $\Lambda = \Lambda_{KLM} = \langle x^K, y^L, z^M\rangle$, the finite-index subgroup of Γ generated by x^K, y^L, and z^M.

Lemma 3.3. *Let $\Lambda = \Lambda_{KLM}$ and suppose that $\{T_\lambda \colon \lambda \in \Lambda\}$ is an action of Λ by continuous affine operators on $\ell^\infty(\Gamma,\mathbb{R})$. If C is a weak*-compact, convex subset of $\ell^\infty(\Gamma,\mathbb{R})$ such that $T_\lambda(C) \subset C$ for every $\lambda \in \Lambda$, then there is a common fixed point $b \in C$ for all the T_λ.*

Proof. Put

$$F_n = \big\{ x^{pK} y^{qL} z^{rM} \colon 0 \leqslant p < n,\ 0 \leqslant q < n,\ 0 \leqslant r < n^2 \big\}.$$

The condition on the powers of z is imposed so that any distortion caused by left multiplication of F_n by a given element $\lambda \in \Lambda$ is eventually small. More precisely, for every $\lambda \in \Lambda$ we have

$$\frac{|\lambda F_n \triangle F_n|}{|F_n|} \to 0 \quad \text{as } n \to \infty,$$

where $\triangle$ denotes symmetric difference and $|\cdot|$ denotes cardinality.

Now fix $b_0 \in C$, and let $b_n = \dfrac{1}{|F_n|}\sum_{\lambda\in F_n} T_\lambda(b_0)$. Then $b_n \in C$ because C is convex. Since C is weak*-compact, there is a subsequence b_{n_j} converging weak* to some $b \in C$. Note that $\sup_{c\in C}\|c\|_\infty < \infty$ by compactness of C. Then since each T_θ is continuous, we get that $T_\theta(b_{n_j}) \to T_\theta(b)$ for every $\theta \in \Lambda$. Furthermore,

$$\|T_\theta(b_{n_j}) - b_{n_j}\|_\infty \leqslant \frac{|\theta F_{n_j} \triangle F_{n_j}|}{|F_n|}\sup_{c\in C}\|c\|_\infty \to 0 \quad \text{as } j \to \infty.$$

It follows that $T_\theta(b) = b$ for all $\theta \in \Lambda$. □

The essential point in the previous proof is that Λ is amenable, and that $\{F_n\}$ forms a Følner sequence.

We call a set $A \subset \Lambda = \Lambda_{KLM}$ *almost invariant* if $|A\lambda \triangle A|$ is finite for every $\lambda \in \Lambda$. Clearly, if A is almost invariant, then so is $A\lambda'$ for every $\lambda' \in \Lambda$.

Lemma 3.4. *Let $A \subset \Lambda = \Lambda_{KLM}$ be an infinite almost invariant subset. Then $\Lambda \setminus A$ is finite.*

Proof. Let $S = \{x^K, x^{-K}, y^L, y^{-L}, z^M, z^{-M}\}$ be a set of generators for Λ. The *Cayley graph* $\mathscr{G}$ of Λ with respect to S has as vertices the elements of Λ, and for every vertex λ and $s \in S$ there is a directed edge from λ to λs. Let E be the union of $As \triangle A$ over $s \in S$, so E is finite. We can therefore enclose E in a 'box' of the form $B = \{x^{iK}y^{kL}z^{lM} : |i|, |j|, |k| \leqslant n\}$. Since A is infinite, choose an $a \in A \setminus B$. Then for every $b \in \Lambda \setminus B$ there is a finite directed path in $\mathscr{G}$ from a to b that avoids B, say with vertices $a, as_1, as_1s_2, \ldots, as_1s_2\ldots s_r = b$, and by the definition of E each of these is in A. Hence, $\Lambda \setminus A \subset B$ and so is finite. □

Second proof of Theorem 3.2, adapted from [39]. Suppose that $h \in \mathbb{Z}\Gamma$ with $h + \mathbb{Z}\Gamma f$ having a finite Γ-orbit in $\mathbb{Z}\Gamma/\mathbb{Z}\Gamma f$. We will prove that $h \in \mathbb{Z}\Gamma f$.

There are $K, L \geqslant 1$ such that $(x^K - 1)h \in \mathbb{Z}\Gamma f$ and $(y^L - 1)h \in \mathbb{Z}\Gamma f$. Then $x^{-K}y^Lx^Ky^{-L} = z^{KL}$ also stabilizes $h + \mathbb{Z}\Gamma f$. Let $M = KL$. Then there are $g_1, g_2, g_3 \in \mathbb{Z}\Gamma$ such that $(x^K - 1)h = g_1 f$, $(y^L - 1)h = g_2 f$, and $(z^M - 1)h = g_3 f$.

Consider the finite-index subgroup $\Lambda = \Lambda_{KLM}$ of Γ as above. For every $\lambda \in \Lambda$ there is a $c(\lambda) \in \mathbb{Z}\Gamma$ such that $(\lambda - 1)h = c(\lambda)f$, and this element is unique since $\mathbb{Z}\Gamma$ has no zero-divisors. Then $c\colon \Lambda \to \mathbb{Z}\Gamma$ is a *cocycle*, that is, it obeys $c(\lambda\lambda') = c(\lambda) + \lambda c(\lambda')$ for all $\lambda, \lambda' \in \Lambda$.

Consider $\mathbb{Z}\Gamma$ as a subset of $\ell^\infty(\Gamma, \mathbb{R})$. We claim that c is a uniformly bounded cocycle, that is, $\sup_{\lambda\in\Lambda}\|c(\lambda)\|_\infty < \infty$. The reason for this is that we can calculate the value of $c(\lambda)$ for arbitrary λ using left shifts of the generators g_i that are sufficiently spread out to prevent large accumulations of coefficients. For example, if $p, q, r \geqslant 1$, then applying the cocycle equation first for powers of x^K, then powers of y^L, and then powers of z^M, we get that

$$\begin{aligned} c(x^{pK}y^{qL}z^{rM}) &= g_1 + x^Kg_1 + \cdots + x^{(p-1)K}g_1 + x^{pK}g_2 + x^{pK}y^Lg_2 + \cdots \\ &\quad + x^{pK}y^{(q-1)L}g_2 + x^{pK}y^{qL}g_3 + x^{pK}y^{qL}z^Mg_3 + \cdots + x^{pK}y^{qL}z^{(r-1)M}g_3. \quad (3.1) \end{aligned}$$

Since the supports of the g_i are finite, there is a uniform bound $P < \infty$ such that for every $\gamma \in \Gamma$ and every $\lambda \in \Lambda$, there are at most P summands in the expression (3.1)

for $c(\lambda)$ whose support contains γ. Hence, $\|c(\lambda)\|_\infty \leqslant P \sup_{1\leqslant i\leqslant 3} \|g_i\|_\infty = B < \infty$, establishing our claim.

Now let C be the closed, convex hull of $\{c(\lambda)\colon \lambda \in \Lambda\}$ in $\ell^\infty(\Gamma, \mathbb{R})$, which is weak*-compact since the $c(\lambda)$ are uniformly bounded. Consider the continuous affine maps $T_\lambda\colon \ell^\infty(\Gamma, \mathbb{R}) \to \ell^\infty(\Gamma, \mathbb{R})$ defined by $T_\lambda(v) = \lambda \cdot v + c(\lambda)$. Then $T_\lambda \circ T_{\lambda'} = T_{\lambda\lambda'}$ by the cocycle property of c, and $T_\lambda(c(\lambda')) = c(\lambda\lambda')$, so that $T_\lambda(C) \subset C$ for all $\lambda \in \Lambda$. By Lemma 3.3 there is a common fixed point $v = (v_\gamma) \in C$ for the T_λ, so that $v - \lambda \cdot v = c(\lambda) \in \mathbb{Z}\Gamma$ for all $\lambda \in \Lambda$. We write each $v_\gamma = w_\gamma + u_\gamma$ with $w_\gamma \in \mathbb{Z}$ and $u_\gamma \in [0, 1)$. Then

$$u_\gamma - u_{\lambda^{-1}\gamma} = v_\gamma - v_{\lambda^{-1}\gamma} + w_\gamma - w_{\lambda^{-1}\gamma} \in (-1, 1) \cap \mathbb{Z} = \{0\},$$

so that w also satisfies $w - \lambda \cdot w = c(\lambda)$, where now $w \in \ell^\infty(\Gamma, \mathbb{Z})$ has integer coordinates. Replacing w by $-w$, we have found a $w \in \ell^\infty(\Gamma, \mathbb{Z})$ with $\|w\|_\infty \leqslant B$ and $\lambda \cdot w - w = c(\lambda)$ for all $\lambda \in \Lambda$.

Next, we use Lemma 3.4 to show that we can replace w by an element of $\ell^\infty(\Gamma, \mathbb{Z})$ having finite support, and so being an element of $\mathbb{Z}\Gamma$. Fix $\gamma \in \Gamma$. For $-B \leqslant k \leqslant B$ consider the 'level set' for the restriction of w to the right coset $\Lambda\gamma$: $A_{\gamma,k} = \{\lambda \in \Lambda\colon w_{\lambda^{-1}\gamma} = k\}$. We claim that for each γ there is exactly one k for which $A_{\gamma,k}$ is infinite. For suppose that $A_{\gamma,k}$ is infinite. Let $\theta \in \Lambda$. Since $\theta \cdot w - w = c(\theta)$ has finite support, $w_{\theta^{-1}\lambda^{-1}\gamma} = w_{\lambda^{-1}\gamma}$ for all but finitely many $\lambda \in \Lambda$. Hence, $|A_{\gamma,k}\theta \triangle A_{\gamma,k}| < \infty$ for every $\theta \in \Lambda$. By Lemma 3.4, we see that $\Lambda \setminus A_{\gamma,k}$ is finite. Thus, for every $\gamma \in \Gamma$ we can adjust the value of w on the coset $\Lambda\gamma$ so that the restriction of w to $\Lambda\gamma$ has finite support. Doing this for each of the finitely many right cosets $\Lambda\gamma$ results in a w with finite support on Γ, so that $w \in \mathbb{Z}\Gamma$.

Thus, $c(\lambda) = (\lambda - 1)w$ for every $\lambda \in \Lambda$. Since $(\lambda - 1)h = c(\lambda)f = (\lambda - 1)wf$, we get that $h = wf \in \mathbb{Z}\Gamma f$, as required.

Theorem 3.2 answers the (1×1)-case of the following natural question.

Problem 3.5. Describe or characterize those $F \in R^{k\times l}$ for which α_F is ergodic, or, equivalently, those Noetherian R-modules M for which α_M is ergodic. Is there a finite algorithm that will decide whether or not a given α_F is ergodic? Are there easily checked sufficient conditions on F for ergodicity of α_F?

Einsiedler and Rindler provided one answer to Problem 3.5, which involves the notion of prime ideals in R. A two-sided ideal $\mathfrak{p}$ in R is *prime* if whenever $\mathfrak{a}$ and $\mathfrak{b}$ are two-sided ideals in R with $\mathfrak{ab} \subseteq \mathfrak{p}$, then either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. If N is an R-submodule of an R-module M, then the annihilator $\operatorname{ann}_R(N)$ of N is defined as $\{f \in R\colon fn = 0 \text{ for all } n \in N\}$, which is a two-sided ideal in R. A prime ideal $\mathfrak{p}$ is *associated* with M if there is a submodule $N \subseteq M$ such that for every non-zero submodule $N' \subseteq N$ we have $\operatorname{ann}_R(N') = \mathfrak{p}$. Every Noetherian R-module has associated prime ideals, and there are only finitely many of them.

Call a prime ideal $\mathfrak{p}$ of R *ergodic* if the subgroup $\{\gamma \in \Gamma\colon \gamma - 1 \in \mathfrak{p}\}$ of Γ has infinite index in Γ. For instance, the ideal $\mathfrak{p}$ in Example 3.7 is prime (being the kernel of a ring homomorphism onto a commutative integral domain), but is not ergodic. It is easy to verify that a prime ideal $\mathfrak{p}$ is ergodic if and only if the action $\alpha_{R/\mathfrak{p}}$ is ergodic, and that the only ideal associated with $R/\mathfrak{p}$ is $\mathfrak{p}$.

Theorem 3.6 ([18], Theorem 3.3). *Let M be a Noetherian R-module. Then α_M is ergodic if and only if every prime ideal associated with M is ergodic.*

Example 3.7. Let $\mathfrak{p}$ be the left ideal in R generated by $x-1$ and $y-1$. Then

$$z-1=(y-z)(x-1)+(1-xz)(y-1)\in\mathfrak{p},$$

and so the map $\phi\colon R\to\mathbb{Z}$ defined by $\phi(f)=f(1,1,1)$ is a well-defined surjective ring homomorphism with kernel $\mathfrak{p}$. Thus, $\mathfrak{p}$ is a prime ideal with $R/\mathfrak{p}$ isomorphic to $\mathbb{Z}$. The dual Γ-action is simply the identity map on $\mathbb{T}$, which is non-ergodic. Hence, $\mathfrak{p}$ is a left ideal generated by two elements with $\alpha_{R/\mathfrak{p}}$ non-ergodic, showing that Theorem 3.2 does not extend to non-principal actions.

We remark that if we consider $\mathbb{Z}^3$ instead of Γ, then the characterization of ergodic prime ideals in [60], Theorem 6.5 shows that their complex variety is finite, and, in particular, they must have at least three generators by elementary dimension theory.

A relatively explicit description of all prime ideals in R is given in [48].

Problem 3.8. Characterize the ergodic prime ideals in R.

An answer to this problem would reduce Problem 3.5 to computing the prime ideals associated with a given Noetherian R-module. However, this appears to be difficult, even for modules of the form R/Rf, although it follows from Theorem 3.2 that all prime ideals associated with R/Rf must be ergodic.

4. Mixing

Let Δ be a countable discrete group, and let M be a left $\mathbb{Z}\Delta$-module. Denote by μ Haar measure on X_M. The associated algebraic Δ-action α_M is called *mixing* if $\lim_{\delta\to\infty}\mu(\alpha_M^\delta(E)\cap F)=\mu(E)\mu(F)$ for every pair of measurable sets $E,F\subset X_M$, where $\delta\to\infty$ refers to the one-point compactification of Δ. For $m\in M$, the *stabilizer of m* is the subgroup $\{\delta\in\Delta\colon \delta\cdot m=m\}$.

Proposition 4.1 ([60], Theorem 1.6). *The algebraic Δ-action α_M is mixing if and only if the stabilizer of every non-zero $m\in M$ is finite. In the case $\Delta=\Gamma$, this is equivalent to requiring that for every non-zero $m\in M$ the map $\gamma\mapsto\gamma\cdot m$ is injective on Γ.*

Using this together with some of the ideas from the previous section, we can give a simple criterion for $g(z)\in\mathbb{Z}Z=\mathbb{Z}[z^{\pm}]$ such that α_g is mixing.

Proposition 4.2. *Let $g=g(z)\in\mathbb{Z}Z$. Then the principal Γ-action α_g is mixing if and only if $g(z)$ has no roots that are roots of unity.*

Proof. Suppose first that $g(z)$ has a root that is a root of unity, so that $g(z)$ has a factor $g_0(z)\in\mathbb{Z}Z$ dividing z^n-1 for some $n\geqslant 1$. Then $h=g/g_0\notin\mathbb{Z}\Gamma g$, but $(z^n-1)h=[(z^n-1)/g_0]g\in\mathbb{Z}\Gamma g$, so that α_g is not mixing by Proposition 4.1.

Conversely, suppose that g has no root that is a root of unity. Recall that the content $\mathsf{c}(h)$ of an element $h=\sum_{i,j}h_{ij}(z)x^iy^j$ is the greatest common divisor in $\mathbb{Z}Z$ of the polynomials $h_{ij}(z)$. Then $h\in\mathbb{Z}\Gamma g$ if and only if $g\mid\mathsf{c}(h)$.

If $(x^p y^q z^r - 1)h \in \mathbb{Z}\Gamma g$ with $(p,q) \neq (0,0)$, then g divides $\mathsf{c}\big((x^p y^q z^r - 1)h\big) = \mathsf{c}(h)$, showing that $h \in \mathbb{Z}\Gamma g$. Similarly, if $(z^r - 1)h \in \mathbb{Z}\Gamma g$, then $g \mid (z^r - 1)\mathsf{c}(h)$, and by assumption g is relatively prime to $z^r - 1$. Hence, again $g \mid \mathsf{c}(h)$, and so $h \in \mathbb{Z}\Gamma g$. Then Proposition 4.1 shows that α_g is mixing. □

Using more elaborate algebra, Hayes found several sufficient conditions on $f \in \mathbb{Z}\Gamma$ for α_f to be mixing. To make the cyclotomic nature of these conditions clear, recall that the nth *cyclotomic polynomial* $\Phi_n(u)$ is given by $\Phi_n(u) = \prod(u - \omega)$, where the product is over all primitive nth roots of unity. Each $\Phi_n(u)$ is irreducible in $\mathbb{Q}[u]$, and $u^n - 1 = \prod_{d|n} \Phi_d(u)$ is the irreducible factorization of $u^n - 1$ in $\mathbb{Q}[u]$.

Let $u_1, \dots, u_r$ be r commuting variables. Then a *generalized cyclotomic polynomial in* $\mathbb{Z}[u_1^{\pm}, \dots, u_r^{\pm}]$ is one of the form $\Phi_k(u_1^{n_1} \cdots u_r^{n_r})$ for some $k \geqslant 1$ and some choice of the integers $n_1, \dots, n_r$, not all equal to 0.

There is a well-defined ring homomorphism $\pi\colon \mathbb{Z}\Gamma \to \mathbb{Z}[\overline{x}^{\pm}, \overline{y}^{\pm}]$, with $\overline{x}$ and $\overline{y}$ commuting variables, given by $x \mapsto \overline{x}$, $y \mapsto \overline{y}$, and $z \mapsto 1$. For $f = \sum_{ij} f_{ij}(z) x^i y^j \in \mathbb{Z}\Gamma$, its image under π is $\overline{f}(\overline{x}, \overline{y}) = \sum_{ij} f_{ij}(1) \overline{x}^i \overline{y}^j$.

Proposition 4.3 [31]. *Each of the following conditions on $f \in \mathbb{Z}\Gamma$ is sufficient for α_f to be mixing*:

1) *$f \in \mathbb{Z}[x^{\pm}, z^{\pm}]$ and f is not divisible by a generalized cyclotomic polynomial in x and z*;

2) *$f \in \mathbb{Z}[y^{\pm}, z^{\pm}]$ and f is not divisible by a generalized cyclotomic polynomial in y and z*;

3) *$f = \sum_{ij} f_{ij}(z) x^i y^j$ with the content $\mathsf{c}(f)$ not divisible by a cyclotomic polynomial in z, and $\overline{f}(\overline{x}, \overline{y})$ not divisible by a generalized cyclotomic polynomial in $\overline{x}$ and $\overline{y}$.*

Example 4.4. (a) If $f = 1 + x + y$, then α_f is mixing by part 3).

(b) If $f = x + z - 2$, then α_f is mixing by part 1), yet $\overline{f}(\overline{x}, \overline{y}) = \overline{x} - 1$ is cyclotomic, showing that part 3) is not necessary.

(c) A generalized cyclotomic polynomial in u_1 and u_2 always has a root both of whose coordinates have absolute value 1. It follows that, for example, $4u_1 + 3u_2 + 8u_1u_2$ cannot be divisible by any generalized cyclotomic polynomial. Then part 3) above implies that for every choice of non-zero integers p, q, r, the polynomial $f = (z^p + 3)x + (z^q + 2)y + (z^r + 7)xy$ yields a mixing α_f.

More generally, if $\sum_{ij} b_{ij} u_1^i u_2^j$ is not divisible by a generalized cyclotomic polynomial, and $p_{ij}(z) \in \mathbb{Z}\mathrm{Z}$ all satisfy $p_{ij}(1) = b_{ij}$ and have no common root that is a root of unity, then $f = \sum_{ij} p_{ij}(z) x^i y^j$ results in a mixing α_f.

Problem 4.5. Does there exist a finite algorithm that decides, given $f \in \mathbb{Z}\Gamma$, whether or not α_f is mixing? More generally, is there such an algorithm that decides mixing for Γ-actions of the form α_F, where $F \in \mathbb{Z}\Gamma^{k \times l}$?

There is another, simply stated, sufficient condition for α_f to be mixing. Recall that $\ell^1(\Gamma, \mathbb{R})$ is a Banach algebra under convolution, with identity element 1.

Proposition 4.6. *If $f \in \mathbb{Z}\Gamma$ is invertible in $\ell^1(\Gamma, \mathbb{R})$, then α_f is mixing.*

Proof. If α_f were not mixing, then there would be an $h \notin \mathbb{Z}\Gamma f$ and an infinite subgroup $\Lambda \subset \Gamma$ such that $\lambda \cdot h - h \in \mathbb{Z}\Gamma f$ for all $\lambda \in \Lambda$, say $\lambda \cdot h - h = c(\lambda) f$. Let $w \in \ell^1(\Gamma, \mathbb{R})$ be the inverse of f. Then $\lambda \cdot h \cdot w - h \cdot w = c(\lambda) \in \mathbb{Z}\Gamma$. Letting $\lambda \to \infty$ in Λ, we see that $g = h \cdot w \in \mathbb{Z}\Gamma$. Hence, $h = h \cdot w \cdot f = gf \in \mathbb{Z}\Gamma f$, showing that α_f is mixing. □

We will see in Theorem 5.1 that invertibility of f in $\ell^1(\Gamma, \mathbb{R})$ corresponds to an important dynamical property of α_f.

5. Expansiveness

Let Δ be a countable discrete group and α be an algebraic Δ-action on a compact Abelian group X. Then α is called *expansive* if there is a neighbourhood U of 0_X in X such that $\bigcap_{\delta \in \Delta} \alpha^\delta(U) = \{0_X\}$. All groups X we consider are metrizable, so let d be a metric on X compatible with its topology. By averaging d over X, we may assume that d is translation-invariant. Then α is expansive provided there is a $\kappa > 0$ such that if $d\big(\alpha^\delta(t), \alpha^\delta(u)\big) \leqslant \kappa$ for all $\delta \in \Delta$, then $t = u$. Any such κ is a so-called *expansive constant* of α (with respect to the given metric d).

Expansiveness is an important and useful property, with many implications. It is therefore crucial to know when algebraic actions are expansive. For principal actions there is a simple criterion.

Theorem 5.1 ([17], Theorem 3.2). *Let Δ be a countable discrete group. For $f \in \mathbb{Z}\Delta$ the following are equivalent*:

1) *the principal algebraic action α_f on X_f is expansive*;
2) *the principal algebraic action α_{f^*} on X_{f^*} is expansive*;
3) *f is invertible in $\ell^1(\Delta, \mathbb{R})$*;
4) *the right convolution $w \mapsto \rho_f(w) = w \cdot f^*$ is injective on $\ell^\infty(\Delta, \mathbb{R})$.*

Before sketching the proof, we isolate a crucial property of $\ell^1(\Delta, \mathbb{R})$ called *direct finiteness*: if $v, w \in \ell^1(\Delta, \mathbb{R})$ with $v \cdot w = 1$, then $w \cdot v = 1$, where $v \cdot w$ denotes the usual convolution product in $\ell^1(\Delta, \mathbb{R})$. This was originally proved by Kaplansky [34], p. 122 using von Neumann algebra techniques. Later Montgomery [51] gave a short proof using C^*-algebra methods. A more self-contained argument using only elementary ideas was given by Passman [55]. All these arguments use a key feature of $\ell^1(\Delta, \mathbb{R})$, that it has a faithful trace function $\mathrm{tr}\colon \ell^1(\Delta, \mathbb{R}) \to \mathbb{R}$ given by $\mathrm{tr}(w) = w_{1_\Delta}$. This function has the properties that it is linear, $\mathrm{tr}(1) = 1$, $\mathrm{tr}(v \cdot w) = \mathrm{tr}(w \cdot v)$, and $\mathrm{tr}(v \cdot v) \geqslant 0$, with equality if and only if $v = 0$. The key argument is that if $e \cdot e = e$, then $0 \leqslant \mathrm{tr}(e) \leqslant 1$, and $\mathrm{tr}(e) = 1$ implies that $e = 1$. If $v \cdot w = 1$, then $e = w \cdot v$ satisfies $e \cdot e = (w \cdot v) \cdot (w \cdot v) = w \cdot (v \cdot w) \cdot v = w \cdot v = e$, and $\mathrm{tr}(w \cdot v) = \mathrm{tr}(v \cdot w) = 1$, hence $w \cdot v = 1$.

Proof of Theorem 5.1. Let $d_{\mathbb{T}}$ be the usual metric on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by $d_{\mathbb{T}}(t + \mathbb{Z}, u + \mathbb{Z}) = \min_{n \in \mathbb{Z}} |t - u + n|$. It is straightforward to check that we may use the pseudometric d_1 on X_f defined by $d_1(t, u) = d_{\mathbb{T}}(t_{1_\Delta}, u_{1_\Delta})$ to define expansiveness (see [15], Proposition 2.3 for details).

First suppose that there is a $w \in \ell^\infty(\Delta, \mathbb{R})$ such that $\rho_f(w) = w \cdot f^* = 0$. Let $\beta\colon \mathbb{R} \to \mathbb{T}$ be the usual projection map, and extend β to $\ell^\infty(\Delta, \mathbb{R})$ coordinatewise. For every $\varepsilon > 0$ we have $\rho_f(\varepsilon w) = 0$, so that $\beta(\varepsilon w) \in X_f$. Since $\|\varepsilon w\|_\infty = \varepsilon \|w\|_\infty$

can be made arbitrarily small, it follows that α_f is not expansive. Conversely, if α_f is not expansive, then there is a point $t \in X_f$ with $d_{\mathbb{T}}(t_\delta, 0) < (3\|f^*\|_1)^{-1}$ for all $\delta \in \Delta$. Pick $\widetilde{t}_\delta \in [0,1)$ with $\beta(\widetilde{t}_\delta) = t_\delta$ for all $\delta \in \Delta$, and let $\widetilde{t} = (\widetilde{t}_\delta) \in \ell^\infty(\Delta, \mathbb{R})$. Then $\rho_f(\widetilde{t}) \in \ell^\infty(\Delta, \mathbb{Z})$), and $\|\rho_f(\widetilde{t})\|_\infty < 1$, hence $\rho_f(\widetilde{t}) = 0$. This shows that 1) $\Leftrightarrow$ 4).

Now suppose that ρ_f is injective on $\ell^\infty(\Delta, \mathbb{R})$. Then $\rho_{f^*}\big(\ell^1(\Delta, \mathbb{R})\big)$ is dense in $\ell^1(\Delta, \mathbb{R})$ by the Hahn–Banach Theorem. Since the set of invertible elements in $\ell^1(\Delta, \mathbb{R})$ is open, there is a $w \in \ell^1(\Delta, \mathbb{R})$ with $w \cdot f^* = 1$. By direct finiteness, f^* is invertible in $\ell^1(\Delta, \mathbb{R})$ with inverse w. Hence, $f^{-1} = (w^*)^{-1}$. This shows that 4) $\Rightarrow$ 3), and the implication 3) $\Rightarrow$ 4) is obvious. Since f is invertible if and only if f^* is invertible, 2) $\Leftrightarrow$ 1). □

Let us call $f \in \mathbb{Z}\Delta$ *expansive* if α_f is expansive. For the case $\Delta = \mathbb{Z}$, Wiener's theorem on invertibility in the convolution algebra $\ell^1(\mathbb{Z}, \mathbb{C})$ shows that $f(u) \in \mathbb{Z}[u^{\pm}]$ is expansive if and only if f does not vanish on $\mathbb{S} = \{c \in \mathbb{C} \colon |c| = 1\}$. The usual proof of Wiener's Theorem via Banach algebras is non-constructive since it uses Zorn's lemma to create maximal ideals. Cohen [12] has given a constructive treatment of this and similar results. We are grateful to David Boyd for showing us a simple algorithm for deciding expansiveness in this case.

Proposition 5.2. *There is a finite algorithm, using only operations in* $\mathbb{Q}[u]$, *that decides, given* $f(u) \in \mathbb{Z}[u^{\pm}]$, *whether or not* f *is expansive.*

Proof. We may assume that $f(u) \in \mathbb{Z}[u]$ with $f(0) \neq 0$, say of degree d. We can compute the greatest common divisor $g(u)$ of $f(u)$ and $u^d f(1/u)$ using only finitely many operations in $\mathbb{Q}[u]$. Observe that any root of $f(u)$ on $\mathbb{S}$ must also be a root of $g(u)$. Let the degree of g be e. Then $g(u) = u^e g(1/u)$, so that the coefficients of $g(u)$ are symmetric. If e is odd, then -1 is a root of g since all other possible roots come in distinct pairs. If e is even, then it is simple to compute an $h(u) \in \mathbb{Q}[u]$ such that $g(u) = u^{e/2} h(u + 1/u)$. Any root of g on $\mathbb{S}$ corresponds to a root of h on $[-2, 2]$. We can then apply Sturm's algorithm, which uses a finite sequence of calculations in $\mathbb{Q}[u]$ and sign changes of rationals, to compute the number of roots of h in $[-2, 2]$. □

Decidability of expansiveness for other groups Δ, even just Γ, is a fascinating open question.

Problem 5.3. Is there a finite algorithm that decides, given $f \in \mathbb{Z}\Gamma$, whether or not f is expansive?

There is one type of polynomial in $\mathbb{Z}\Delta$ that is easily seen to be expansive. Call an $f \in \mathbb{Z}\Delta$ *lopsided* if there is a $\delta_0 \in \Delta$ such that $|f_{\delta_0}| > \sum_{\delta \neq \delta_0} |f_\delta|$. This terminology is due to Purbhoo [57].

If f is lopsided with dominant coefficient f_{δ_0}, then adjust f by multiplying by $\pm\delta_0^{-1}$ so that $f_1 > \sum_{\delta \neq 1} |f_\delta|$. Then $f = f_1(1 - g)$, where $\|g\|_1 < 1$. We can then invert f in $\ell^1(\Delta, \mathbb{R})$ by a geometric series:

$$f^{-1} = \frac{1}{f_1}(1 + g + g * g + \cdots) \in \ell^1(\Delta, \mathbb{R}).$$

Thus, lopsided polynomials are expansive.

The product of two lopsided polynomials need not be lopsided (expand $(3 + u + u^{-1})^2$). Surprisingly, if $f \in \mathbb{Z}\Delta$ is expansive, then there is always a $g \in \mathbb{Z}\Delta$ such that fg is lopsided. This was first proved by Purbhoo [57] for $\Delta = \mathbb{Z}^d$ using a rather complicated induction from the case $\Delta = \mathbb{Z}$, but his methods also provided quantitative information he needed to approximate algorithmically the complex amoeba[2] of a Laurent polynomial in several variables. The following short proof is due to Li.

Proposition 5.4. *Let Δ be a countable discrete group and let $f \in \mathbb{Z}\Delta$ be expansive. Then there is a $g \in \mathbb{Z}\Delta$ such that fg is lopsided.*

Proof. Since f is expansive, by Theorem 5.1 there is a $w \in \ell^1(\Delta, \mathbb{R})$ such that $f \cdot w = 1$. There is an obvious extension of the definition of lopsidedness to $\ell^1(\Delta, \mathbb{R})$. Note that lopsidedness is an open condition in $\ell^1(\Delta, \mathbb{R})$. First perturb w slightly to a w' having finite support, and then again slightly to a w'' having finite support and rational coordinates. This results in a $w'' \in \mathbb{Q}\Delta$ such that fw'' is lopsided. Choose an integer n so that $g = nw'' \in \mathbb{Z}\Delta$. Then fg is lopsided. □

One consequence of the previous result is that if $f \in \mathbb{Z}\Delta$ is expansive, then the coefficients of f^{-1} must decay exponentially fast. Recall that if Δ is finitely generated, then a choice of a finite symmetric generating set S induces the *word norm* $|\cdot|_S$, where $|\delta|_S$ is the length of the shortest word in generators in S whose product is δ. Clearly, $|\delta_1\delta_2|_S \leqslant |\delta_1|_S|\delta_2|_S$. A different choice S' for a symmetric generating set gives an equivalent word norm $|\cdot|_{S'}$ in the sense that there are two constants $c_1, c_2 > 0$ such that $c_1|\delta|_S \leqslant |\delta|_{S'} \leqslant c_2|\delta|_S$ for all $\delta \in \Delta$.

Proposition 5.5. *Let Δ be a finitely generated group, and fix a finite symmetric generating set S. Suppose that $f \in \mathbb{Z}\Delta$ is invertible in $\ell^1(\Delta, \mathbb{R})$. Then there are constants $C > 0$ and $0 < r < 1$ such that $|(f^{-1})_\delta| \leqslant C\, r^{|\delta|_S}$ for all $\delta \in \Delta$.*

Proof. By the previous proposition, there is a $g \in \mathbb{Z}\Delta$ such that $h = fg$ is lopsided. We may assume that the dominant coefficient of h occurs at 1_Δ, so that $h = q(1 - b)$, where $q \in \mathbb{Z}$ and $b \in \mathbb{Q}\Delta$ has $\|b\|_1 = s < 1$. Let $F = \operatorname{supp}(h)$ and put $\tau = \max\{|\delta|_S \colon \delta \in F\}$. Now $h^{-1} = q^{-1}(1 + b + b^2 + \cdots)$ and $\|b^k\|_1 \leqslant \|b\|_1^k \leqslant s^k$ for all $k \geqslant 1$. Furthermore, if $(b^n)_\delta \neq 0$, then $|\delta|_S \leqslant n\tau$. Hence if $|\delta|_S > n\tau$, then $(1 + b + b^2 + \cdots + b^n)_\delta = 0$. Thus,

$$|(h^{-1})_\delta| = |q^{-1}(b^{n+1} + b^{n+2} + \cdots)_\delta| \leqslant q^{-1} \sum_{k=n+1}^{\infty} \|b^k\|_1 \leqslant \frac{q^{-1}}{1 - r} s^{n+1},$$

whenever $|\delta|_S > n\tau$. This shows that $(h^{-1})_\delta \leqslant Cr^{|\delta|_S}$ with $r = s^{1/\tau}$ and suitable $C > 0$. Since $f^{-1} = gh^{-1}$, we obtain the required result with the same r and a different C. □

We remark that a different proof of this proposition, using functional analysis and under stricter hypotheses on Δ, was given in [17], Proposition 4.7. If $\Delta = \mathbb{Z}$ and $f(u) \in \mathbb{Z}[u^{\pm}]$ is invertible in $\ell^1(\mathbb{Z}, \mathbb{R})$, then f does not vanish on $\mathbb{S}$. Then $1/f$

[2] *Russian editor's note*: The *amoeba* of a complex polynomial is the image of its zero set in $(\mathbb{C}^*)^n$ under the map $\operatorname{Log}\colon (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$.

is holomorphic in some annular region around $\mathbb{S}$ in $\mathbb{C}$, and so the coefficients of its Laurent expansion decay exponentially fast, giving a direct proof of the proposition in this case.

Next we focus on the case $\Delta = \Gamma$ and obstructions to invertibility coming from representations of Γ.

Let $\mathscr{H}$ be a complex Hilbert space, and let $\mathscr{B}(\mathscr{H})$ be the algebra of bounded linear operators on $\mathscr{H}$. Denote by $\mathscr{U}(\mathscr{H})$ the group of unitary operators on $\mathscr{H}$. An *irreducible unitary representation* of Γ is a homomorphism $\pi\colon \Gamma \to \mathscr{U}(\mathscr{H})$ for some complex Hilbert space $\mathscr{H}$ such that there is no non-trivial closed subspace of $\mathscr{H}$ invariant under all the operators $\pi(\gamma)$. Then π extends to an algebraic homomorphism $\pi\colon \ell^1(\Gamma, \mathbb{C}) \to \mathscr{B}(\mathscr{H})$ by

$$\pi\Big(\sum_{\gamma} f_\gamma \gamma\Big) = \sum_{\gamma} f_\gamma \pi(\gamma).$$

If there is a non-zero $v \in \mathscr{H}$ with $\pi(f)v = 0$, then clearly f cannot be invertible in $\ell^1(\Gamma, \mathbb{R})$. The converse is also true.

Theorem 5.6. *Let $f \in \mathbb{Z}\Gamma$. Then f is not invertible in $\ell^1(\Gamma, \mathbb{R})$ if and only if there is an irreducible unitary representation $\pi\colon \Gamma \to \mathscr{U}(\mathscr{H})$ on some complex Hilbert space $\mathscr{H}$ and a non-zero vector $v \in \mathscr{H}$ such that $\pi(f)v = 0$.*

This result is stated in [18], Theorem 8.2, and a detailed proof is given in [24], Theorem 3.2. A key element of the proof is that $\ell^1(\Gamma, \mathbb{C})$ is *symmetric*, that is, $1 + g^*g$ is invertible for every $g \in \ell^1(\Gamma, \mathbb{C})$, and, in particular, for every $g \in \mathbb{Z}\Gamma$. There are examples of countable amenable groups whose group algebras are not symmetric [32].

We remark that Theorem 5.6 remains valid when Γ is replaced by any nilpotent group, in particular by $\mathbb{Z}$ or $\mathbb{Z}^d$. In the latter cases all irreducible unitary representations are 1-dimensional, and are obtained by evaluation at a point in $\mathbb{S}^d$. This is exactly the Wiener criterion for invertibility in $\ell^1(\mathbb{Z}^d)$.

The usefulness of Theorem 5.6 is at best limited, since Γ is not of Type I and so its representation theory is murky. However, there is an extension of the Gelfand theory, called Allan's local principle, that detects invertibility in the non-commutative Banach algebra $\ell^1(\Gamma, \mathbb{C})$. The use of this principle for algebraic actions was initiated in [24]. In the case of Γ-actions this principle has an explicit and easily verified form, which we now describe.

To simplify the notation, let $B = \ell^1(\Gamma, \mathbb{C})$ be the complex convolution Banach algebra of Γ, and let $C = \ell^1(\mathrm{Z}, \mathbb{C})$ be its center. The maximal ideals of C all have the form $\mathfrak{m}_\zeta = \big\{v = \sum_{i=-\infty}^{\infty} v_j z^j \colon v(\zeta) = 0\big\}$, where $\zeta \in \mathbb{S}$. For $v \in C$ the quotient norm of $v + \mathfrak{m}_\zeta \in C/\mathfrak{m}_\zeta$ is easily seen to be $\|v + \mathfrak{m}_\zeta\|_{C/\mathfrak{m}_\zeta} = |v(\zeta)|$. Let $\mathfrak{b}_\zeta$ denote the two-sided ideal in B generated by $\mathfrak{m}_\zeta$, so that

$$\mathfrak{b}_\zeta = \Big\{w = \sum_{i,j} w_{ij}(z) x^i y^j \in B \colon w_{ij}(z) \in \mathfrak{m}_\zeta \text{ for all } i, j \in \mathbb{Z}\Big\}.$$

Then for $w = \sum_{i,j} w_{ij}(z)x^i y^j \in B$, the quotient norm of $w + \mathfrak{b}_\zeta \in B/\mathfrak{b}_\zeta$ is

$$\|w + \mathfrak{b}_\zeta\|_{B/\mathfrak{b}_\zeta} = \sum_{i,j=-\infty}^{\infty} |w_{ij}(\zeta)|.$$

To give a concrete realization of $B/\mathfrak{b}_\zeta$, we introduce variables U, V, subject to the skew commutativity relation $VU = \zeta UV$. Then the skew-commutative convolution Banach algebra $\ell^1_\zeta(U,V)$ consists of all sums $\sum_{i,j=-\infty}^{\infty} c_{ij}U^iV^j$ with $c_{ij} \in \mathbb{C}$ and $\sum_{i,j} |c_{ij}| < \infty$, and with multiplication the usual convolution modified by skew commutativity. Then the map $B/\mathfrak{b}_\zeta \to \ell^1_\zeta(U,V)$ given by

$$\sum_{i,j} w_{ij}(z)x^i y^j + \mathfrak{b}_\zeta \mapsto \sum_{i,j} w_{ij}(\zeta)U^iV^j$$

is an isometric isomorphism of complex Banach algebras. Under identification of these algebras, the quotient map $\pi_\zeta \colon B \to B/\mathfrak{b}_\zeta \cong \ell^1_\zeta(U,V)$ takes the concrete form

$$\pi_\zeta\left(\sum_{i,j} w_{ij}(z)x^i y^j\right) = \sum_{i,j} w_{ij}(\zeta)U^iV^j.$$

In [24], § 3 *Allan's local principle* was introduced as a convenient device for checking expansiveness of principal actions of Γ and then applied to a number of examples.

Theorem 5.7 (Allan's local principle, [2]). *An element $w \in \ell^1(\Gamma, \mathbb{C})$ is invertible if and only if $\pi_\zeta(w)$ is invertible in $\ell^1_\zeta(U,V)$ for every $\zeta \in \mathbb{S}$.*

Proof. In the notation introduced above, it is clear that if w is invertible in B, then all its homomorphic images $\pi_\zeta(w)$ must also be invertible.

Conversely, assume that $w \in B$ is non-invertible but $\pi_\zeta(w)$ is invertible in $\ell^1_\zeta(U,V)$ for every $\zeta \in \mathbb{S}$. The left ideal Bw is proper since w is not invertible, so let $\mathfrak{b}$ be a maximal left ideal in B containing Bw. It is easy to see that $\mathfrak{b} \cap C$ must be a maximal ideal in C, hence $\mathfrak{b} \cap C = \mathfrak{m}_\zeta$ for some $\zeta \in \mathbb{S}$. Then $\mathfrak{b}_\zeta \subseteq \mathfrak{b}$. By assumption, $\pi_\zeta(w) = w + \mathfrak{b}_\zeta$ is invertible in $B/\mathfrak{b}_\zeta$, and $w \in \mathfrak{b}$, so that $\mathfrak{b} = B$, a contradiction. □

We now turn to some concrete examples where expansiveness can be analyzed geometrically. But before doing so, let us introduce a quantity that we will use extensively.

Definition 5.8. Let $f(u) \in \mathbb{Z}[u^{\pm}]$. The *logarithmic Mahler measure* of f is defined as

$$\mathsf{m}(f) = \int_{\mathbb{S}} \log|f(\xi)|\, d\xi.$$

The *Mahler measure* of f is defined as $\mathsf{M}(f) = \exp(\mathsf{m}(f))$.

Suppose that $f(u) = c_n u^n + \cdots + c_u + c_0$ with $c_n c_0 \neq 0$, and factor $f(u)$ over $\mathbb{C}$ as $c_n \prod_{j=1}^n (u - \lambda_j)$. Then Jensen's formula shows that $\mathsf{m}(f)$ has the alternative

expression

$$\mathsf{m}(f) = \log|c_n| + \sum_{|\lambda_j|>1} \log|\lambda_j| = \log|c_n| + \sum_{j=0}^{n} \log^+|\lambda_j|, \tag{5.1}$$

where $\log^+ r = \max\{\log r, 0\}$ for $r \geqslant 0$ (cf. [49]).

Mahler's motivation was to derive important inequalities in transcendence theory. Using the fact that $\mathsf{M}(fg) = \mathsf{M}(f)\mathsf{M}(g)$, he showed that if $f, g \in \mathbb{C}[u]$, then $\|fg\|_1 \geqslant 2^{-\deg f - \deg g}\|f\|_1\|g\|_1$, and that the constant is best possible.

Let us consider the case of an $f \in \mathbb{Z}\Gamma$ that is linear in y, so that $f(x,y,z) = h(x,z)y - g(x,z)$. We will find 1-dimensional α_f-invariant subspaces of $\ell^\infty(\Gamma,\mathbb{C})$ as follows. Let $\Lambda = \langle x, z\rangle$ be the subgroup of Γ generated by x and z. To make calculations in $\ell^\infty(\Gamma,\mathbb{C})$ and $\ell^\infty(\Lambda,\mathbb{C})$ more transparent, we will write elements as formal sums $w = \sum_{\gamma\in\Gamma} w(\gamma)\gamma$ and $v = \sum_{\lambda\in\Lambda} v(\lambda)\lambda$.

Fix $(\xi,\zeta) \in \mathbb{S}^2$ and define

$$v_{\xi,\zeta} = \sum_{k,m=-\infty}^{\infty} \xi^k\zeta^m\, x^k z^m \in \ell^\infty(\Lambda,\mathbb{C}). \tag{5.2}$$

Observe that

$$\rho_x(v_{\xi,\zeta}) = \Big(\sum_{k,m} \xi^k\zeta^m\, x^k z^m\Big)x^{-1} = \xi \sum_{k,m} \xi^k\zeta^m\, x^k z^m = \xi\, v_{\xi,\zeta}$$

and, similarly, $\rho_z(v_{\xi,\zeta}) = \zeta\, v_{\xi,\zeta}$. Hence, the 1-dimensional subspace $\mathbb{C}v_{\xi,\zeta}$ is a common eigenspace for the operators ρ_x and ρ_z. It follows that $\rho_q(v_{\xi,\zeta}) = v_{\xi,\zeta}\, q^*(x,z) = q(\xi,\zeta)v_{\xi,\zeta}$ for every $q \in \mathbb{Z}\Lambda$.

Let $\{c_n\}$ be a sequence of complex constants to be determined, and consider the point $w = \sum_{n=-\infty}^{\infty} c_n\, v_{\xi,\zeta}\, y^n$. With use of the relations $y^k q(x,z) = q(xz^k,z)y^k$ for all $k \in \mathbb{Z}$ and all $q \in \mathbb{Z}\Lambda$, the condition $\rho_f(w) = w \cdot f^* = 0$ becomes

$$\begin{aligned} 0 &= \Big(\sum_{n=-\infty}^{\infty} c_n\, v_{\xi,\zeta}\, y^n\Big)\big(y^{-1}h^*(x,z) - g^*(x,z)\big) \\ &= \sum_{n=-\infty}^{\infty} \big\{c_n\, v_{\xi,\zeta}\, h^*(xz^{n-1},z)y^{n-1} - c_n\, v_{\xi,\zeta}\, g^*(xz^n,z)y^n\big\} \\ &= \sum_{n=-\infty}^{\infty} \big\{c_{n+1}h(\xi\zeta^n,\zeta) - c_n g(\xi\zeta^n,\zeta)\big\}v_{\xi,\zeta}\, y^n. \end{aligned}$$

This calculation shows that $\rho_f(w) = 0$ if and only if the c_n satisfy

$$c_{n+1}h(\xi\zeta^n,\zeta) = c_n g(\xi\zeta^n,\zeta) \quad \text{for all } n \in \mathbb{Z}. \tag{5.3}$$

Since $\|w\|_\infty = \sup_{n\in\mathbb{Z}}|c_n|$, one way to create non-expansive elements f of this form is to find conditions on g and h that guarantee the existence of a non-zero bounded solution $\{c_n\}$ of (5.3) for some choice of ξ and ζ.

Suppose that both g and h do not vanish on $\mathbb{S}^2$. Fix a non-zero value of c_0. Then by (5.3), the other values of c_n are determined by

$$c_n = \begin{cases} c_0 \displaystyle\prod_{j=0}^{n-1} \frac{g(\xi\zeta^j,\zeta)}{h(\xi\zeta^j,\zeta)} & \text{for } n \geqslant 1, \\ c_0 \displaystyle\prod_{j=1}^{-n} \left[\frac{g(\xi\zeta^{-j},\zeta)}{h(\xi\zeta^{-j},\zeta)}\right]^{-1} & \text{for } n \leqslant -1. \end{cases} \tag{5.4}$$

Let

$$\phi_\zeta(\xi) = \log\left|\frac{g(\xi,\zeta)}{h(\xi,\zeta)}\right| \tag{5.5}$$

and consider the map $\psi_\zeta(n,\xi)\colon \mathbb{Z}\times\mathbb{R}\to\mathbb{R}$ given by

$$\psi_\zeta(n,\xi) = \begin{cases} \displaystyle\sum_{j=0}^{n-1} \phi_\zeta(\xi\zeta^j) & \text{for } n \geqslant 1, \\ 0 & \text{for } n = 0, \\ \displaystyle -\sum_{j=1}^{-n} \phi_\zeta(\xi\zeta^{-j}) & \text{for } n \leqslant -1. \end{cases} \tag{5.6}$$

Then ψ_ζ satisfies the cocycle equation

$$\psi_\zeta(m+n,\xi) = \psi_\zeta(m,\xi) + \psi_\zeta(n,\xi\zeta^m)$$

for all $m,n\in\mathbb{Z}$ and $\xi\in\mathbb{S}$. Furthermore, there is a non-zero bounded solution $\{c_n\}$ of (5.3) if and only if $\{\psi_\zeta(n,\xi)\colon -\infty < n < \infty\}$ is bounded above.

Suppose that $\zeta\in\mathbb{S}$ is irrational, and that there is a $\xi\in\mathbb{S}$ for which (5.3) has a non-zero bounded solution. Observe that ϕ_ζ is continuous on $\mathbb{S}^2$ since neither g nor h vanish there by assumption. By the ergodic theorem,

$$\int_{\mathbb{S}} \phi_\zeta(\xi)\,d\xi = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \phi_\zeta(\xi\zeta^j) = \lim_{n\to\infty} \frac{1}{n}\psi_\zeta(n,\xi) \leqslant 0,$$

$$-\int_{\mathbb{S}} \phi_\zeta(\xi)\,d\xi = \lim_{n\to\infty} -\frac{1}{n}\sum_{j=1}^{n} \phi_\zeta(\xi\zeta^{-j}) = \lim_{n\to\infty} \frac{1}{n}\psi_\zeta(-n,\xi) \leqslant 0.$$

Thus, $\int_{\mathbb{S}} \phi_\zeta(\xi)\,d\xi = \mathsf{m}\big(g(\,\cdot\,,\zeta)\big) - \mathsf{m}\big(h(\,\cdot\,,\zeta)\big) = 0$. There is a similar necessary condition when ζ is rational, but here the integral is replaced by a finite sum over a coset of the finite orbit of ζ in $\mathbb{S}$. It turns out that these necessary conditions are also sufficient.

Theorem 5.9. *Let $f(x,y,z) = h(x,z)y - g(x,z) \in \mathbb{Z}\Gamma$, and suppose that both g and h do not vanish anywhere on $\mathbb{S}^2$. Let $\phi_\zeta(\xi) = \log|g(\xi,\zeta)/h(\xi,\zeta)|$. Then α_f is expansive if and only if*

1) *$\int_{\mathbb{S}} \phi_\zeta(\xi)\,d\xi \neq 0$ for every irrational $\zeta\in\mathbb{S}$, and*
2) *$\sum_{j=0}^{n-1} \phi_\zeta(\xi\zeta^j) \neq 0$ for every nth root of unity $\zeta\in\mathbb{S}$ and every $\xi\in\mathbb{S}$.*

Proof. Suppose first that α_f is non-expansive. By Theorem 5.1, the Γ-invariant subspace $K = \{w \in \ell^\infty(\Gamma, \mathbb{C})\colon \rho_f(w) = 0\}$ is non-trivial. By restricting the Γ-action on K to the commutative subgroup Λ, we can apply the argument in [60], Lemma 6.8 to find a 1-dimensional Λ-invariant subspace $W \subset K$. In other words, there are a non-zero $w \in K$ and points $\xi, \zeta \in \mathbb{S}$ such that $\rho_x(w) = \xi w$ and $\rho_z(w) = \zeta w$. It follows from this and the above discussion that w must have the form $w = \sum_{n=-\infty}^{\infty} c_n\, v_{\xi,\zeta} y^n$ with the $c_n \in \mathbb{C}$ given by (5.3) and with $\{c_n\}$ a bounded sequence. If ζ is an nth root of unity, then clearly $\sum_{j=0}^{n-1} \phi_\zeta(\xi\zeta^j) = 0$ by boundedness of the c_n, while if ζ is irrational, then the discussion above shows that $\int_{\mathbb{S}} \phi_\zeta(\xi)\, d\xi = 0$.

For the converse, suppose first that ζ is rational, say $\zeta^k = 1$. If there is a $\xi \in \mathbb{S}$ with $\sum_{j=0}^{k-1} \phi_\zeta(\xi\zeta^{-j}) = 0$, then (5.4) and (5.5) show that there is a bounded sequence $\{c_n\}$, in this case even periodic with period k, such that $w = \sum c_n\, v_{\xi,\zeta}\, y^n \in \ell^\infty(\Gamma, \mathbb{C})$ with $\rho_f(w) = 0$, showing that α_f is non-expansive.

Finally, suppose that ζ is irrational and $\int_{\mathbb{S}} \phi_\zeta(\xi)\, d\xi = 0$. If there were a continuous coboundary b on $\mathbb{S}$ such that $\phi_\zeta(\xi) = b(\xi\zeta) - b(\xi)$, then the cocycle $\psi_\zeta(\,\cdot\,, \xi)$ would be bounded for every value of ξ, and, as before, this means we can form a non-zero point $w \in \ell^\infty(\Gamma, \mathbb{C})$ with $\rho_f(w) = 0$, so α_f is non-expansive. Thus, suppose that no such coboundary exists. Let $Y = \mathbb{S} \times \mathbb{R}$, and define a skew product transformation $S\colon Y \to Y$ by $S(\xi, r) = (\xi\zeta, r + \phi_\zeta(\xi))$. Since ϕ_ζ is not a coboundary, but $\int_{\mathbb{S}} \phi_\zeta(\xi)\, d\xi = 0$, the homeomorphism S is topologically transitive (see [27] or [4], Theorems 1 and 2). By [6], pp. 38-39, there exists a point $\xi \in \mathbb{S}$ such that the entire S orbit of $(\xi, 0)$ has its second coordinate bounded above. Since $S^n(\xi, 0) = (\xi\zeta^n, \psi_\zeta(n, \xi))$, it follows that for this choice of ξ, the sequence $\{c_n\}$ given by (5.4) and (5.5) is also bounded. Then for $w = \sum_{n=-\infty}^{\infty} c_n\, v_{\xi,\zeta} y^n \in \ell^\infty(\Gamma, \mathbb{C})$ we have $\rho_f(w) = 0$, and so α_f is non-expansive. □

Remark 5.10. With the assumptions of Theorem 5.9 that neither g nor h vanish on $\mathbb{S}^2$, the functions $\mathsf{m}\big(g(\,\cdot\,, \zeta)\big)$ and $\mathsf{m}\big(h(\,\cdot\,, \zeta)\big)$ are both continuous functions of ζ. Hence, the conditions 1) and 2) in Theorem 5.9 combine to say that the graphs of these functions never cross (for rational ζ use the Mean Value Theorem). In particular, if $h(x, z) \equiv 1$, then since $\int_{\mathbb{S}} \mathsf{m}\big(g(\,\cdot\,, \zeta)\big)\, d\zeta = \mathsf{m}(g) \geqslant 0$, the condition for expansiveness of $f(x, y, z) = y - g(x, z)$ becomes simply that $\mathsf{m}\big(g(\,\cdot\,, \zeta)\big) > 0$ for all ζ.

Example 5.11. The polynomial $f(x, y, z) = 3 + x + y + z$, although not lopsided, was shown to be expansive in [18], Example 7.4. In [24], Example 3.6, four different ways to verify its expansiveness are given: 1) using irreducible unitary representations, 2) direct computation of its inverse in $\ell^1(\Gamma, \mathbb{R})$, 3) using Allan's local principle, and 4) using the geometric argument in Theorem 5.9.

Many more examples illustrating various aspects of expansiveness (or its lack) for polynomials in $\mathbb{Z}\Gamma$ are contained in [24].

Example 5.12 ([24], Example 5.11). Let $f(x, y, z) = x + y + z + 2 = y - g(x, z) \in \mathbb{Z}\Gamma$, where $g(x, z) = -x - z - 2$. Since $g(-1, -1) = 0$, g does not quite satisfy the hypotheses of Theorem 5.9. However, we can directly use the violation of 2) there to show that α_f is non-expansive.

Let $\zeta_0 = -1$. We want to find $\xi \in \mathbb{S}$ such that $|g(\xi,-1)g(-\xi,-1)| = 1$. This amounts to solving the equation $|\xi^2 - 1| = 1$, which has the four solutions $\pm e^{\pm \pi i/6}$. Let $\xi_0 = e^{\pi i/6}$ and consider the point $v_{\xi_0,-1}$ as defined in (5.2). Then the coefficients c_n of the point $\sum_{n=-\infty}^{\infty} c_n v_{\xi_0,-1} y^n \in \ker \rho_f$ satisfy (5.3), and are hence alternately multiplied by $g(\xi_0,-1)$ and by $g(-\xi_0,-1)$, where $|g(\xi_0,-1)| = \sqrt{2-\sqrt{3}} \cong 0.51764$ and $|g(-\xi_0,-1)| = 1/|g(\xi_0,-1)| \cong 1.93185$. Thus, $\{c_n\}_{n=-\infty}^{\infty}$ is bounded, and hence α_f is non-expansive.

Note that here $\mathsf{m}(g(\,\cdot\,,\zeta)) = \log|\zeta+2|$, which vanishes only at $\zeta = -1$. Hence, for all irrational ζ, condition 1) of Theorem 5.9 is satisfied. It is an easy exercise to show that if $\zeta \neq -1$ is rational, then condition 2) is also satisfied. So here the only values of (ξ,ζ) leading to a bounded solution $\{c_n\}$ of (5.3) are $(\pm e^{\pm \pi i/6}, -1)$. This example appears as Example 10.6 in [18], but was incorrectly asserted there to be expansive.

Example 5.13. Let $f(x,y,z) = y^2 - xy - 1 \in \mathbb{Z}\Gamma$. For $(\xi,\zeta) \in \mathbb{S}^2$ let $v_{\xi,\zeta}$ be defined by (5.2), and let $w = \sum_{n=-\infty}^{\infty} c_n\, v_{\xi,\zeta} y^n$. Although f is now quadratic in y, we can still calculate $\rho_f(w)$ as before, finding that $\rho_f(w) = 0$ if and only if

$$c_{n+2} = \xi\, \zeta^n c_{n+1} + c_n \quad \text{for all } n \in \mathbb{Z}. \tag{5.7}$$

Thus, we need conditions on (ξ,ζ) for which (5.7) has a bounded solution.

For $n \geqslant 1$ put

$$A_n(\xi,\zeta) = \begin{bmatrix} 0 & 1 \\ 1 & \xi\zeta^{n-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \xi\zeta^{n-2} \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & \xi \end{bmatrix}. \tag{5.8}$$

Then the recurrence relation (5.7) shows that

$$A_n(\xi,\zeta) \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} c_n \\ c_{n+1} \end{bmatrix} \quad \text{for all } n \geqslant 1,$$

and there is a similar formula for $n \leqslant -1$.

We are therefore reduced to finding pairs (ξ,ζ) such that the matrix-valued cocycle $\{A_n(\xi,\zeta)\}$ is bounded. The easiest place to look is the case $\zeta = 1$, since $A_n(\xi,1) = A_1(\xi,1)^n$. Now $A_1(\xi,1)$ has eigenvalues on $\mathbb{S}$ only if the roots of $u^2 - \xi u - 1 = 0$ are there. This happens exactly when $\xi = \pm i$. Hence if we define c_n by

$$\begin{bmatrix} c_n \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & i \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{for all } n \in \mathbb{Z},$$

then $w = \sum_{k,n,m=-\infty}^{\infty} c_n i^k\, x^k y^n z^m \in \ell^\infty(\Gamma,\mathbb{C}) \cap \ker(\rho_f)$, so α_f is non-expansive.

For some values of (ξ,ζ) the growth rate of $A_n(\xi,\zeta)$ can be strictly positive, for example $(1,1)$. However, as we will see later (see (9.7)), the growth rate of $A_n(\xi,\zeta)$ as $n \to \pm\infty$ is zero for almost every $(\xi,\zeta) \in \mathbb{S}^2$, although this is not at all obvious.

If g or h are allowed to vanish on $\mathbb{S}^2$, then there is a completely different source of non-expansive behavior.

Example 5.14. Let $f(x,y,z) = h(x,z)y - g(x,z) \in \mathbb{Z}\Gamma$, and suppose that $g(x,z)$ and $h(xz^{-1},z)$ have a common zero $(\xi,\zeta) \in \mathbb{S}^2$. Consider $v_{\xi,\zeta}$ as a point in $\ell^\infty(\Gamma,\mathbb{C})$. Then

$$\begin{aligned}\rho_f(v_{\xi,\zeta}) &= v_{\xi,\zeta} \cdot \big(y^{-1}h^*(x,z) - g^*(x,z)\big) = v_{\xi,\zeta} \cdot \big(h^*(xz^{-1},z)y^{-1} - g^*(x,z)\big)\\ &= h(\xi\zeta^{-1},\zeta)v_{\xi,\zeta} \cdot y^{-1} - g(\xi,\zeta)v_{\xi,\zeta} = 0,\end{aligned}$$

so that ρ_f is not expansive.

There is a simple geometric method to create examples in this situation. Start with two polynomials $g(x,z)$ and $h_0(x,z)$ whose unitary varieties in $\mathbb{S}^2$ do not intersect (cf. the definition on p. 683), but which have the property that the unitary variety of the skewed polynomial $h_0(xz^m,z)$ intersects that of g for sufficiently large m. An example of this form was described in [25] (Example 8.5), which also contains other results about expansiveness in cases when at least one unitary variety is non-empty. However, this analysis leaves open one very interesting case. The simplest version of this question is the following.

Problem 5.15. Let $g(x,z) \in \mathbb{Z}[x^{\pm},z^{\pm}]$, and suppose that there is a $\zeta \in \mathbb{S}$ for which $\mathsf{m}\big(g(\,\cdot\,,\zeta)\big) = 0$ and such that $g(\xi,\zeta)$ vanishes for at least one value of $\xi \in \mathbb{S}$. Is there a value of ξ for which the partial sums $\sum_{j=0}^{n} \log|g(\xi\zeta^j,\zeta)|$ are bounded above for all n?

Frączek and Lemańczyk showed in [20] that these sums are unbounded for almost every $\xi \in \mathbb{S}$. The argument of Besicovitch [6] to prove the existence of bounded sums makes essential use of continuity, and does not apply in this case where there are logarithmic singularities.

Remark 5.16. By invoking some deep results in diophantine approximation theory, we can show that the second alternative in the last paragraph of the proof of Theorem 5.9 never occurs. For we claim that if $\int_{\mathbb{S}} \phi_\zeta(\xi)\,d\xi = 0$, then ζ must be an algebraic number in $\mathbb{S}$. Assuming this for the moment, a result of Gelfond [23] then shows that for every $\varepsilon > 0$ there is a $C > 0$ such that $|\zeta^n - 1| > Ce^{-n\varepsilon}$ for every $n \geqslant 1$. Since $\phi_\zeta(\xi)$ is smooth, its Fourier coefficients $\widehat{\phi_\zeta}(n)$ decay rapidly as $|n| \to \infty$, and therefore the sequence $b_n = \widehat{\phi_\zeta}(n)/(\zeta^n - 1)$ for $n \neq 0$ also decays rapidly, so that the formal solution $b(\xi) = \sum_{-\infty}^{\infty} b_n\xi^n$ of the equation $\phi_\zeta(\xi) = b(\xi\zeta) - b(\xi)$ is continuous.

To justify our assertion, we write

$$g(x,\zeta) = A(\zeta)\prod_{j=1}^{m}(x - \lambda_j(\zeta)) \quad \text{and} \quad h(x,\zeta) = B(\zeta)\prod_{k=1}^{n}(x - \mu_k(\zeta)),$$

where $A(z)$ and $B(z)$ are Laurent polynomials with integer coefficients, and the λ_j and μ_k are algebraic functions. The condition $\int_{\mathbb{S}} \phi_\zeta(\xi)\,d\xi = 0$ becomes, via Jensen's formula,

$$\left|A(\zeta)\prod_{|\lambda_j(\zeta)|>1}\lambda_j(\zeta)\right| = \left|B(\zeta)\prod_{|\mu_k(\zeta)|>1}\mu_k(\zeta)\right|,$$

which is an algebraic equation in ζ, so ζ is algebraic.

From this and the preceding proof, under our assumptions that neither g nor h vanish anywhere on $\mathbb{S}^2$, we conclude that if $\int_{\mathbb{S}} \phi_\zeta(\xi)\, d\xi = 0$ and ζ is irrational, then *every* choice of ξ will yield a non-zero point $w = \sum c_n\, v_{\xi,\zeta}\, y^n \in \ell^\infty(\Gamma, \mathbb{C})$ with $\rho_f(w) = 0$.

This idea was observed independently by Evgeny Verbitskiy (oral communication).

To illustrate some of the preceding ideas, we provide an informative example. This was chosen so that the diophantine estimates mentioned in Remark 5.16 can be given an elementary and self-contained proof, rather than appealing to difficult general diophantine results. In addition, the constants in our analysis are effective enough to rule out non-expansive behavior at all rational ζ. One consequence is that for this algebraic Γ-action, non-expansiveness cannot be detected by looking at only finite-dimensional representations of Γ.

Example 5.17. Let

$$g(x, z) = a(x)c(z), \quad \text{where } a(x) = x^2 - x - 1 \text{ and } c(z) = z^{12} + z^2 + 1.$$

It is easy to check that neither a nor c vanishes on $\mathbb{S}$, so that g does not vanish anywhere on $\mathbb{S}^2$. Also, $a(x)$ has roots $\tau = (1+\sqrt{5}\,)/2$ and $\sigma = (1-\sqrt{5}\,)/2 = -\tau^{-1}$.

Consider $f(x, y, z) = y - g(x, z)$, so that here $h(x, z) \equiv 1$. Then $\phi_\zeta(\xi) = \log|g(\xi, \zeta)| = \log|a(\xi)| + \log|c(\zeta)|$. Define $\psi_\zeta(n, \xi)$ as in (5.6), and say that $(\xi, \zeta) \in \mathbb{S}^2$ is *non-expansive for* g if the sequence $\{\psi_\zeta(n, \xi)\colon n \in \mathbb{Z}\}$ is bounded.

We claim that there are eight values $\zeta_1, \dots, \zeta_8 \in \mathbb{S}$ which are algebraically conjugate algebraic integers of degree 48 such that the non-expansive points for g are exactly those of the form (ξ, ζ_k), where ξ is any element of $\mathbb{S}$ and $1 \leqslant k \leqslant 8$.

We start with two simple results.

Lemma 5.18. *Suppose that $\zeta \in \mathbb{S}$ is irrational, and that there are constants $C > 0$ and $0 < r < 1$ such that $|\zeta^n - 1| \geqslant Cr^n$ for all $n \geqslant 1$. If $\kappa \in \mathbb{C}$ with $|\kappa| < r$, then the function $\xi \mapsto \log|1 - \xi\kappa|$ on $\mathbb{S}$ is a continuous coboundary for ζ, that is, there is a continuous function $b\colon \mathbb{S} \to \mathbb{R}$ such that $\log|1 - \xi\kappa| = b(\xi\zeta) - b(\xi)$.*

Proof. Since $|\xi\kappa| < 1$, the Taylor series for $\log(1 - \xi\kappa)$ converges, and

$$\log(1 - \xi\kappa) = -\sum_{n=1}^{\infty} \frac{\xi^n \kappa^n}{n}\,.$$

Let $B(\xi) = \sum_{n=1}^{\infty} b_n \xi^n$. Then $B(\xi\zeta) - B(\xi) = \log(1 - \xi\kappa)$ provided that $b_n = -\kappa^n/[(\zeta^n - 1)n]$ for all $n \geqslant 1$. The assumption on ζ implies that $|b_n| \leqslant C^{-1}(|\kappa|/r)^n$, so that the series for B converges uniformly on $\mathbb{S}$. Then $b(\xi) = \mathrm{Re}\{B(\xi)\}$ gives the required coboundary. □

Lemma 5.19. *Let $p(u) \in \mathbb{Z}[u]$ be monic and irreducible. Suppose that p has a root $\zeta \in \mathbb{S}$ that is irrational. Then there is a constant $C > 0$ such that*

$$|\zeta^n - 1| \geqslant C\mathsf{M}(p)^{-n/2} \quad \textit{for all } n \geqslant 1, \tag{5.9}$$

where $\mathsf{M}(p) > 1$ is the Mahler measure of p.

Proof. We factor p over $\mathbb{C}$ as

$$p(u) = (u - \zeta)(u - \overline{\zeta}) \prod_{j=1}^{r} (u - \lambda_j),$$

where $\overline{\zeta}$ is the complex conjugate of ζ. The polynomial

$$p^{(n)}(u) = (u - \zeta^n)(u - \overline{\zeta}^n) \prod_{j=1}^{r} (u - \lambda_j^n)$$

has integer coefficients, and $p^{(n)}(1) \neq 0$, hence $|p^{(n)}(1)| \geqslant 1$. Then using the trivial estimates that $|\lambda^n - 1| \leqslant 2$ if $|\lambda| \leqslant 1$ and $|\lambda^n - 1| \leqslant (1 - 1/|\lambda|)|\lambda^n|$ if $|\lambda| > 1$, we get that

$$\begin{aligned} 1 \leqslant |p^{(n)}(1)| &= |\zeta^n - 1|\,|\overline{\zeta}^n - 1| \prod_{j=1}^{r} |\lambda_j^n - 1| \leqslant |\zeta^n - 1|^2 \cdot 2^r \prod_{|\lambda_j|>1} |\lambda_j^n - 1| \\ &\leqslant |\zeta^n - 1|^2 \cdot 2^r \left\{ \prod_{|\lambda_j|>1} \left(1 - \frac{1}{|\lambda_j|}\right) \right\} \mathsf{M}(p)^n. \end{aligned}$$

Hence, (5.9) is valid with

$$C = 2^{-r/2} \prod_{|\lambda_j|>1} \left(1 - \frac{1}{|\lambda_j|}\right)^{-1/2}. \qquad \square$$

Recall that $a(x) = (x - \tau)(x - \sigma)$, so that $\mathsf{m}(a) = \log \tau$. By Theorem 5.9, we need to find just those $\zeta \in \mathbb{S}$ for which

$$0 = \int_{\mathbb{S}} \log |g(\xi, \zeta)|\, d\xi = \int_{\mathbb{S}} \big(\log |a(\xi)| + \log |c(\zeta)|\big)\, d\xi = \log \tau + \log |c(\zeta)|,$$

that is, we must solve the equation $|c(\zeta)| = \tau^{-1}$. Any such solution $\zeta \in \mathbb{S}$ would also satisfy $c(z)c(1/z) = \tau^{-2}$, or, equivalently, satisfy

$$F(z) = z^{12}\big(c(z)c(z^{-1}) - \tau^{-2}\big) \in \mathbb{Q}(\tau)[z].$$

Here $F(z)$ has degree 24, and has eight roots $\zeta_1, \ldots, \zeta_8 \in \mathbb{S}$. To show that these are algebraic integers, multiply $F(z)$ by its Galois conjugate over $\mathbb{Q}(\tau)$, obtaining the polynomial

$$\begin{aligned} G(z) = {} & z^{48} + 2z^{46} + z^{44} + 2z^{38} + 5z^{36} + 5z^{34} + 2z^{32} + z^{28} + 5z^{26} \\ & + 7z^{24} + 5z^{22} + z^{20} + 2z^{16} + 5z^{14} + 5z^{12} + 2z^{10} + z^4 + 2z^2 + 1, \end{aligned}$$

whose irreducibility is confirmed by *Mathematica*. Hence, the ζ_k are conjugate algebraic integers of degree 48, as claimed.

Outside the unit circle G has 10 roots, whose product is $\mathsf{M}(G) \cong 1.90296$. Then $\sqrt{\mathsf{M}(G)} \cong 1.37948 < \tau$, which plays a crucial role in dealing with rational ζ.

By Theorem 5.1, the only irrational ζ we need to consider are the eight numbers ζ_k. By Lemma 5.19, there is a constant $C > 0$ such that $|\zeta_k^n - 1| > Cr^n$, where $r = \mathsf{M}(G)^{-1/2} < 1$. Since $\tau^{-1} = |\sigma| < r$, by Lemma 5.18 we can find continuous coboundaries b_1 and b_2 such that

$$\log|1 - \tau^{-1}\xi| = b_1(\xi\zeta_k) - b_1(\xi) \quad \text{and} \quad \log|\xi - \sigma| = b_2(\xi\zeta_k) - b_2(\xi).$$

Hence,

$$\psi_{\zeta_k}(\xi) = \log|(\xi - \tau)(\xi - \sigma)c(\zeta_k)| = \log|1 - \tau^{-1}\xi| + \log|\xi - \sigma|$$

is also a coboundary. Thus, (ξ, ζ_k) is non-expansive for every $\xi \in \mathbb{S}$. This is an example of the algebraic phenomenon discussed in Remark 5.16.

To complete our analysis, we turn to the rational case, say $\zeta = \omega$, a primitive nth root of unity. The idea of the following argument is to show that there is a number N_0 large enough such that for all $n \geqslant N_0$ the variation of the n-periodic function $\prod_{j=0}^{n-1} a(\xi\omega^j)$ is small compared to $|c(\omega)|$. The estimates are sharp enough to obtain the bound $N_0 = 143$, and the remaining cases with $n < 143$ can be checked by hand.

First observe that

$$\begin{aligned}\prod_{j=0}^{n-1}\{a(\xi\omega^j)\tau^{-1}\} &= \prod_{j=0}^{n-1}\frac{\xi\omega^j - \tau}{\tau}(\xi\omega^j - \sigma) = (-1)^n\prod_{j=0}^{n-1}(1 - \tau^{-1}\xi\omega^j)(\xi\omega^j - \sigma)\\ &= (-\xi^n)(1 - \tau^{-n}\xi^n)(1 - \sigma^n\xi^{-n}).\end{aligned}$$

Since $\big|\log|1 - \kappa|\big| \leqslant 2|\kappa|$ for all $\kappa \in \mathbb{C}$ with $|\kappa| \leqslant \tau^{-1}$, it follows that

$$\left|\log\left|\prod_{j=0}^{n-1} a(\xi\omega^j)\tau^{-1}\right|\right| \leqslant 4\tau^{-n} \tag{5.10}$$

for all $\xi \in \mathbb{S}$ and all $n \geqslant 1$. Recalling that $c(\zeta_k) = \tau^{-1}$, we obtain

$$\log\left|\prod_{j=0}^{n-1} a(\xi\omega^j)c(\omega)\right| = \log\left|\prod_{j=0}^{n-1} a(\xi\omega^j)\tau^{-1}\right| + \log\left|\left[\frac{c(\omega)}{c(\zeta_k)}\right]^n\right|.$$

Thus, if

$$\big|n\log|c(\omega)\tau|\big| > 5\tau^{-n}, \tag{5.11}$$

then we must have $\sum_{j=0}^{n-1}\phi_{\zeta_k}(\xi\omega^j) \neq 0$ for all $\xi \in \mathbb{S}$.

To obtain a reasonable bound for N_0 so that (5.11) holds for all $n \geqslant N_0$, we need to make a more careful estimate for the polynomial $G(z)$ than in the proof of Lemma 5.19. Here $G(z)$ has ten roots $\lambda_1, \dots, \lambda_{10}$ outside the unit disk. Thus, besides these and ζ and $\overline{\zeta}$, there are 36 roots of G on or inside the unit circle. The estimate in the proof of Lemma 5.19 can be refined as follows:

$$1 \leqslant |\zeta_k^n - 1|^2 \cdot 2^{36}\prod_{j=1}^{10}|\lambda_j^n - 1| = |\zeta_k^n - 1|^2 \cdot 2^{36}\mathsf{M}(G)^n\prod_{j=1}^{10}\left|1 - \frac{1}{\lambda_j^n}\right|.$$

It is easy to check that $\prod_{j=1}^{10} |1 - \lambda_j^{-n}|$ has a maximum of about 37.94 at $n = 6$. Hence,

$$|\zeta_k^n - 1| \geqslant \frac{1}{2^{18}\sqrt{40}} \mathsf{M}(G)^{-n/2} \quad \text{for all } n \geqslant 1.$$

Since $\zeta_k^n - 1 = \zeta_k^n - \omega^n = (\zeta_k - \omega)(\zeta_k^{n-1} + \zeta_k^{n-2}\omega + \cdots + \omega^{n-1})$, it follows that $|\zeta_k - \omega| \geqslant \frac{1}{n}|\zeta_k^n - 1|$.

The verification of (5.11) breaks into two cases, depending on whether or not ω is close to some ζ_k. Let $\varepsilon_0 = 0.01$. It is an exercise in calculus to show that if $|\omega - \zeta_k| > \varepsilon_0$ for every k, then $\big|\log|c(\omega)\tau|\big| \geqslant \varepsilon_0$, while if $|e^{2\pi i s} - \zeta_k| < \varepsilon_0$ for some k, then the derivative of $\log|c(e^{2\pi i s})\tau|$ has absolute value $\geqslant 1$. A glance at Fig. 1 should make clear the meaning of these statements.

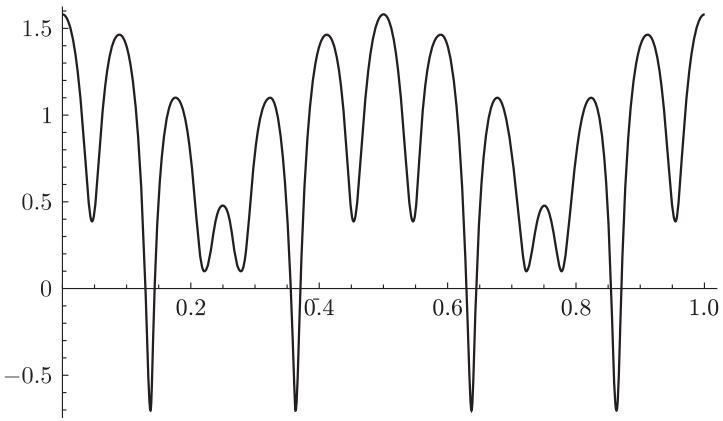

Figure 1. Graph of $\log|c(e^{2\pi i s})\tau|$.

In the first case, the inequality (5.11) is satisfied if $n\varepsilon_0 > 5\tau^{-n}$, which is true for all $n \geqslant 8$.

In the second case, using the lower bound on the absolute value of the derivative, we have

$$\begin{aligned}\big|\log|c(\omega)\tau|\big| &= \big|\log|c(\omega)| - \log|c(\zeta_k)|\big| \\ &\geqslant |\omega - \zeta_k| \geqslant \frac{1}{n}|\zeta_k^n - 1| \geqslant \frac{1/n}{2^{18}\sqrt{40}} \mathsf{M}(G)^{-n/2}.\end{aligned}$$

Since $\mathsf{M}(G)^{1/2} < \tau$, the last term is eventually greater than $5\tau^{-n}$, and in fact this holds for all $n \geqslant 143$. One can then check by hand that (5.11) holds for all $n < 143$, completing our analysis of this example.

It is sometimes useful to make a change of variables in order to transform a polynomial into a form that is easier to analyze (see for example [44], (3)–(6)). Let Δ be a countable discrete group, and let $\mathrm{aut}(\Delta)$ denote the group of automorphisms of Δ. If $\Phi \in \mathrm{aut}(\Delta)$, then Φ will act on various objects associated with Δ. For example, if $f = \sum_\delta f_\delta \delta \in \mathbb{Z}\Delta$, then $\Phi f = \sum_\delta f_\delta \Phi(\delta)$, and so $\Phi(fg) = \Phi(f)\Phi(g)$

and $\Phi(f^*) = \Phi(f)^*$. Analogous formulae hold for $\mathbb{T}^\Delta$, $\ell^1(\Delta, \mathbb{R})$, and $\ell^\infty(\Delta, \mathbb{R})$. If α is an action of Δ, then there is a new Δ-action α^Φ defined by $(\alpha^\Phi)^\delta = \alpha^{\Phi(\delta)}$.

Lemma 5.20. *Let $\Phi \in \mathrm{aut}(\Delta)$ and $f \in \mathbb{Z}\Delta$. Then Φ induces a continuous group isomorphism $\Phi\colon X_f \to X_{\Phi f}$ intertwining the Δ-actions α_f and α_f^Φ, so that (X_f, α_f) and $(X_{\Phi f}, \alpha_f^\Phi)$ are topologically conjugate Δ-actions.*

Proof. If $t \in \mathbb{T}^\Delta$, then

$$t \in X_f \Leftrightarrow t \cdot f^* = 0 \Leftrightarrow 0 = \Phi(t \cdot f^*) = \Phi(t) \cdot \Phi(f)^* \Leftrightarrow \Phi(t) \in X_{\Phi f},$$

so that Φ induces the required isomorphism. Since $\Phi(\delta \cdot t) = \Phi(\delta) \cdot t$, we see that Φ intertwines α_f and α_f^Φ. □

Remark 5.21. It is important to emphasize what Lemma 5.20 does *not* say. Though α_f^Φ and $\alpha_{\Phi f}$ are both Δ-actions on $X_{\Phi f}$, there is no obvious relation between them, even if Δ is commutative. For example, if $\Delta = \mathbb{Z} \cong \langle u \rangle$, $f(u) = u^2 - u - 1$, and $\Phi(u) = u^{-1}$, then $(X_{\Phi f}, \alpha_{\Phi f})$ is conjugate to the $\mathbb{Z}$-action of $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ on $\mathbb{T}^2$, while $(X_{\Phi f}, \alpha_f^\Phi)$ is conjugate to the $\mathbb{Z}$-action of A^{-1} on $\mathbb{T}^2$. But A and A^{-1} do not even have the same eigenvalues, so cannot give algebraically conjugate $\mathbb{Z}$-actions.

However, certain dynamical properties are clearly shared between α_f and $\alpha_{\Phi f}$, for example ergodicity, mixing, and expansiveness.

Automorphisms of Γ have an explicit description.

Lemma 5.22. *Every automorphism of Γ is uniquely determined by integers a, b, c, d, r, and s with $ad - bc = \pm 1$, and is given by $\Phi(x) = x^a y^b z^r$, $\Phi(y) = x^c y^d z^s$, and $\Phi(z) = z^{ad-bc}$.*

Proof. Clearly, Φ induces an automorphism of $\Gamma/\mathrm{Z} \cong \mathbb{Z}^2$, hence $ad - bc = \pm 1$. The condition $\Phi(yx) = \Phi(xyz)$ shows that $\Phi(z) = z^{ad-bc}$. These necessary conditions are easily checked to also be sufficient. □

An analysis of expansiveness for non-principal actions has been carried out by Chung and Li [11]. Let $F \in \mathbb{Z}\Gamma^{k\times k}$ be a square matrix over $\mathbb{Z}\Gamma$. Then $\mathbb{Z}\Gamma^k/\mathbb{Z}\Gamma^k F$ is a left $\mathbb{Z}\Gamma$-module whose dual gives an algebraic action α_F on X_F. An argument similar to the proof of the implication 3) ⇒ 1) in Theorem 5.1 shows that if F has an inverse in $\ell^1(\Gamma, \mathbb{R})^{k\times k}$, then α_F is expansive on X_F. The converse is also true, and leads to a description of all expansive Γ-actions.

Theorem 5.23 ([11], Theorem 3.1). *Let $F \in \mathbb{Z}\Gamma^{k\times k}$ and let α_F be the associated algebraic Γ-action on X_F. Then α_F is expansive if and only if F is invertible in $\ell^1(\Gamma, \mathbb{R})^{k\times k}$. Moreover, every expansive Γ-action is isomorphic to the restriction of such an α_F to a closed α_F-invariant subgroup of X_F.*

Example 5.24. Let $F = \begin{bmatrix} 2 & x \\ y & 2 \end{bmatrix} \in \mathbb{Z}\Gamma^{2\times 2}$. Simply by formally solving the equations for the inverse matrix, one arrives at

$$F^{-1} = \begin{bmatrix} 2(4-xy)^{-1} & -x(4-yx)^{-1} \\ -y(4-xy)^{-1} & 2(4-yx)^{-1} \end{bmatrix},$$

where the inverses appearing in the entries are all in $\ell^1(\Gamma,\mathbb{R})$ by lopsidedness.

Obviously, an algorithm for invertibility of square matrices would immediately answer Problem 5.3. But if the answer to the latter is affirmative, would this provide an algorithm to decide invertibility of square matrices?

Problem 5.25. Suppose that there is an algorithm which decides whether or not an element in $\mathbb{Z}\Gamma$ has an inverse in $\ell^1(\Gamma,\mathbb{R})$. Is there then an algorithm that decides, given $F \in \mathbb{Z}\Gamma^{k\times k}$, whether or not F has an inverse with entries in $\ell^1(\Gamma,\mathbb{R})$?

If α is an expansive algebraic Δ-action on X, and Y is a closed α-invariant subgroup, then clearly the restriction of α to Y is also expansive. However, the question of whether the quotient action $\alpha_{X/Y}$ on X/Y is expansive is much more difficult. When $\Delta = \mathbb{Z}^d$, expansiveness of the quotient is always true ([59], Theorem 3.11), but the proof uses commutative algebra and is not dynamical. Chung and Li conjecture [11] that quotients of expansive actions are always expansive for nilpotent groups Δ. Even for the Heisenberg group this is not known.

Problem 5.26. If α is an expansive action of Γ on a compact Abelian group X, and if Y is a closed α-invariant subgroup of X, then must the quotient action of α on X/Y be expansive?

6. Homoclinic points

Let Δ be a countable discrete group and α an algebraic Δ-action on a compact Abelian group X. An element $t \in X$ is called *homoclinic for* α, or simply *homoclinic*, if $\alpha^\delta(t) \to 0_X$ as $\delta \to \infty$. The set of homoclinic points forms a subgroup of X called the *homoclinic group of* α. The established notation in the literature for this group is $\Delta_\alpha(X)$. It will always be clear from the context (and a slight font change) what Δ (or Δ) refers to.

Homoclinic points are an important technical device for localizing the behavior of points in the group. For example, they are used to construct periodic points, to prove a strong orbit tracing property called specification, and to estimate entropy. They are also a natural starting point for constructing symbolic covers of algebraic actions.

For $\Delta = \mathbb{Z}^d$, many properties of homoclinic groups were studied in detail in [41], especially for principal actions. Let us briefly describe some of the main results there, with a view to extensions to Γ.

For $f \in \mathbb{Z}\mathbb{Z}^d = \mathbb{Z}[u_1^{\pm},\dots,u_d^{\pm}]$ we define the *complex variety* of f to be

$$\mathsf{V}(f) = \big\{(z_1,\dots,z_d) \in (\mathbb{C}^\times)^d \colon f(z_1,\dots,z_d) = 0\big\},$$

where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, and also the *unitary variety* of f to be

$$\mathsf{U}(f) = \big\{(z_1, \dots, z_d) \in \mathsf{V}(f) \colon |z_1| = \cdots = |z_d| = 1\big\}.$$

By Theorem 5.1 and Wiener's theorem, α_f is then expansive if and only if $\mathsf{U}(f) = \varnothing$. In this case, let $w^\Delta = (f^*)^{-1} \in \ell^1(\mathbb{Z}^d)$. As before, let the map $\beta \colon \ell^\infty(\Delta, \mathbb{R}) \to \mathbb{T}^\Delta$ be given by $(\beta w)_\delta = w_\delta \pmod 1$, which clearly commutes with the left Δ-actions. Put $t^\Delta = \beta(w^\Delta)$. Since $\rho_f(t^\Delta) = \beta(\rho_f(w^\Delta)) = \beta(w \cdot f^*) = \beta(1) = 0$, we see that the point t^Δ is in X_f and is also homoclinic. Furthermore, t^Δ is *fundamental*, in the sense that every homoclinic point is a finite integral combination of translates of t^Δ (cf. [41], Lemma 4.5). In this case all homoclinic points decay rapidly enough to have summable coordinates.

In order to describe homoclinic points of principal Δ-actions α_f, we first 'linearize' X_f as follows. Put

$$W_f = \beta^{-1}(X_f) = \big\{w \in \ell^\infty(\Delta, \mathbb{R}) \colon \rho_f(w) \in \ell^\infty(\Delta, \mathbb{Z})\big\}.$$

Suppose now that $f \in \mathbb{Z}\Delta$ is expansive, and define $w^\Delta = (f^*)^{-1} \in \ell^1(\Delta, \mathbb{R})$. Then ρ_f is invertible on $\ell^\infty(\Delta, \mathbb{R})$, and $W_f = \rho_f^{-1}\big(\ell^\infty(\Delta, \mathbb{Z})\big)\big)$, where $\rho_f^{-1}(u) = u \cdot w^\Delta$ for every $u \in \ell^\infty(\Delta, \mathbb{Z})$.

Proposition 6.1 ([17], Propositions 4.2 and 4.3). *Let Δ be a countable discrete group, and let $f \in \mathbb{Z}\Delta$ be expansive, so that f is invertible in $\ell^1(\Delta, \mathbb{R})$. Put $w^\Delta = (f^*)^{-1}$ and let $\pi \colon \ell^\infty(\Delta, \mathbb{Z}) \to X_f$ be defined as $\pi(u) = \beta(u \cdot w^\Delta)$, where β is reduction of coordinates* (mod 1). *Then*:

1) *$\pi \colon \ell^\infty(\Delta, \mathbb{Z}) \to X_f$ is surjective, and in fact the restriction of π to the set of those u with $\|u\|_\infty \leqslant \|f\|_1$ is also surjective*;
2) *$\ker \pi = \rho_f\big(\ell^\infty(\Delta, \mathbb{Z})\big)$*;
3) *π commutes with the relevant left Δ-actions*; *and*
4) *π is continuous in the weak* topology on closed, bounded subsets of $\ell^\infty(\Delta, \mathbb{Z})$.*

Proof. Suppose that $t \in X_f$. There is a unique lift $\widetilde{t} \in \ell^\infty(\Delta, \mathbb{R})$ with $\beta(\widetilde{t}) = t$ and $\widetilde{t}_\delta \in [0, 1)$ for all $\delta \in \Delta$. Then $\beta\big(\rho_f(\widetilde{t})\big) = \rho_f\big(\beta(\widetilde{t})\big) = \rho_f(t) = 0$ in X_f, hence $\rho_f(\widetilde{t}) \in \ell^\infty(\Delta, \mathbb{Z})$, and in fact $\|\rho_f(\widetilde{t})\|_\infty \leqslant \|f\|_1$. Furthermore,

$$\pi\big(\rho_f(\widetilde{t})\big) = \beta\big(\widetilde{t} \cdot f^* \cdot w^\Delta\big) = \beta\big(\widetilde{t} \cdot f^* \cdot (f^*)^{-1}\big) = t.$$

This proves 1), and the remaining parts are routine verifications. □

If $f \in \mathbb{Z}\Delta$ is expansive, then let $t^\Delta = \beta(w^\Delta)$ and call t^Δ the *fundamental homoclinic point* of α_f. This name is justified by the following.

Proposition 6.2. *Let Δ be a countable discrete group, and let $f \in \mathbb{Z}\Delta$ be expansive. Put $w^\Delta = (f^*)^{-1} \in \ell^1(\Delta, \mathbb{R})$ and $t^\Delta = \beta(w^\Delta) \in \Delta_{\alpha_f}(X_f)$. Then every element of $\Delta_{\alpha_f}(X_f)$ is a finite integral combination of left translates of t^Δ.*

Proof. Suppose that $t \in \Delta_{\alpha_f}(X_f)$, and lift t to an element $\widetilde{t} \in \ell^\infty(\Delta, \mathbb{R})$ as in the proof of Proposition 6.1. Then $\rho_f(\widetilde{t}) \in \ell^\infty(\Delta, \mathbb{Z})$, and since $\widetilde{t}_\delta \to 0$ as $\delta \to \infty$, the coordinates of $\rho_f(\widetilde{t})$ must vanish outside a finite subset of Δ, that is, $\rho_f(\widetilde{t}) = g \in \mathbb{Z}\Delta$. Then $t = \pi\big(\rho_f(\widetilde{t})\big) = \pi(g) = \beta(g \cdot w^\Delta) = g \cdot t^\Delta$ has the required form. □

Next we show that expansive principal actions have a very useful orbit tracing property called specification.

Proposition 6.3 ([17], Proposition 4.4). *Let Δ be a countable discrete group, and let $f \in \mathbb{Z}\Delta$ be expansive. Then for every $\varepsilon > 0$ there is a finite subset K_ε of Δ such that if F_1 and F_2 are arbitrary subsets of Δ with $K_\varepsilon F_1 \cap K_\varepsilon F_2 = \varnothing$ and if $t^{(1)}$ and $t^{(2)}$ are arbitrary points in X_f, then there is a $t \in X_f$ such that $d_{\mathbb{T}}(t_\delta, t^{(i)}_\delta) < \varepsilon$ for every $\delta \in F_i$, $i = 1, 2$.*

Sketch of proof. Let $\varepsilon > 0$. The set K_ε is chosen so that $\sum_{\delta \notin K_\varepsilon} |w^{\Delta}_\delta| < \varepsilon / \|f\|_1$. Lift each $t^{(i)}$ to $\widetilde{t}^{(i)}$, and then truncate each $\rho_f(\widetilde{t}^{(i)})$ to a $u^{(i)}$ having finite support in $K_\varepsilon F_i$. It is then easy to verify that $t = \pi\big(\rho_f(u^{(1)}) + \rho_f(u^{(2)})\big)$ has the required properties (see [17], Proposition 4.4 for details).

A point $t \in X_f$ with $\sum_{\delta \in \Delta} d_{\mathbb{T}}(t_\delta, 0) < \infty$ is called *summable*. Let $\Delta^1_{\alpha_f}(X_f)$ denote the group of all summable homoclinic points for α_f. Summability is crucial in using homoclinic points for dynamical purposes.

Example 6.4. Let $f(u_1, u_2) = 3 - u_1 - u_1^{-1} - u_2 - u_2^{-1} \in \mathbb{Z}[u_1^{\pm}, u_2^{\pm}]$. It is shown in [41], Example 7.3 that $\Delta_{\alpha_f}(X_f)$ is uncountable (indeed, the Fourier series of every smooth density on $\mathsf{U}(f)$ decays to 0 at infinity, and so gives a homoclinic point), but $\Delta^1_{\alpha_f}(X_f) = \{0\}$. Despite their large number, the non-summable homoclinic points are essentially useless here.

Summable homoclinic points may still exist for non-expansive actions. For example, consider $f(u_1, u_2) = 2 - u_1 - u_2$. The formal inverse w of f^* via geometric series is well-defined and has coordinates decaying to 0 at infinity, so that $\beta(w)$ is homoclinic, but the decay is so slow that w is not summable (see [41], Example 7.2). Define a function $F\colon \mathbb{T}^2 \to \mathbb{C}$ by putting $F(s_1, s_2) = f^*(e^{2\pi i s_1}, e^{2\pi i s_2})$. Then $1/F$ is integrable on $\mathbb{T}^2$, and w is just the Fourier transform of $1/F$. Now $1/F$ has a singularity at $(0, 0)$, and we can try to cancel this by multiplying it by a sufficiently high power N of another polynomial $G(s_1, s_2) = g(e^{2\pi i s_1}, e^{2\pi i s_2})$ that also vanishes at $(0, 0)$ so that G^N/F has an absolutely convergent Fourier series, resulting in a summable homoclinic point $g^N \cdot w = g^N/f^*$. For this example, taking $g(u_1, u_2) = u_1 - 1$, we see from the detailed analysis in [42], § 5 that $N = 3$ is the smallest power such that G^N/F has an absolutely convergent Fourier series, providing a summable homoclinic point for α_f.

This 'multiplier method' can be generalized to all $f \in \mathbb{Z}[u_1^{\pm}, \dots, u_d^{\pm}]$ provided that the dimension of $\mathsf{U}(f) \subset \mathbb{S}^d$ is at most $d - 2$. More precisely, with this condition there is another polynomial $g \in \mathbb{Z}[u_1^{\pm}, \dots, u_d^{\pm}]$, not a multiple of f, such that $\mathsf{U}(f) \subset \mathsf{U}(g)$. The corresponding quotient G^N/F has an absolutely convergent Fourier series for sufficiently large N [43], and hence α_f has summable homoclinic points. However, if $\dim \mathsf{U}(f) = d - 1$, then this method fails, and in fact there are no non-zero summable homoclinic points ([43], Theorem 3.2).

Let us turn to a consideration of homoclinic points for principal actions of Γ. If $f \in \mathbb{Z}\Gamma$ is expansive, then we have already seen in Proposition 6.2 how to describe $\Delta_{\alpha_f}(X_f)$, and that this group agrees with $\Delta^1_{\alpha_f}(X_f)$.

Consider $f(x,y,z) = 2 - x - y \in \mathbb{Z}\Gamma$. If $\omega = e^{2\pi i/3}$, then f is in the kernel of the algebra homomorphism $\ell^1(\Gamma,\mathbb{C}) \to \mathbb{C}$ given by $x \mapsto \omega$, $y \mapsto \omega^2$, and $z \mapsto 1$. Hence, f is not expansive.

However, it is shown in [26] that the formal inverse of f can be smoothed by using the multiplier $(z-1)^2$ to create a summable homoclinic point for α_f. The proof uses highly non-trivial combinatorial arguments, starting with the non-commutative expansion

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_z x^k y^{n-k}, \quad \text{where} \quad \begin{bmatrix} n \\ k \end{bmatrix}_z = \prod_{j=0}^{k} \frac{z^{n-j}-1}{z^{j+1}-1}.$$

For $f(x,y,z) = 4 - x - x^{-1} - y - y^{-1}$ the authors of [24] state that they have shown that the multiplier $(z-1)^{12}$ results in a summable homoclinic point, and they conjecture that the multiplier $(z-1)^2$ actually suffices.

For more complicated non-expansive polynomials in $\mathbb{Z}\Gamma$, it is not at all clear what could substitute for the condition on the dimension of the unitary variety in the commutative case.

Problem 6.5. For $f \in \mathbb{Z}\Gamma$, determine explicitly both $\Delta_{\alpha_f}(X_f)$ and $\Delta^1_{\alpha_f}(X_f)$.

Anticipating the entropy material in the next section, we remark that Chung and Li ([11], Theorem 1.1), generalizing our earlier work for $\mathbb{Z}^d$ in [41], showed that $\Delta_{\alpha_f}(X_f) \neq \{0\}$ if and only if α_f has positive entropy, and that $\Delta_{\alpha_f}(X_f)$ is dense in X_f if and only if α_f has completely positive entropy.

For expansive Δ-actions α_f, Proposition 6.1, 1) gives a continuous, equivariant, and surjective map π from the full Δ-shift with symbols $\{-\|f\|_1, -\|f\|_1 + 1, \ldots, \|f\|_1\}$ to X_f, which allows us to view this shift space as a symbolic cover of X_f. In 1992 Vershik showed [67] that for certain hyperbolic toral automorphisms, this symbolic cover could be pruned to a shift of finite type for which the covering map π is one-to-one almost everywhere. This provided an arithmetic approach to the construction of Markov partitions, which were originally found geometrically by Adler and Weiss, and are one of the main motivations for symbolic dynamics. Vershik's arithmetic construction was further investigated in [35], [61]–[63].[3] Einsiedler and the second author [19] considered the problem of extending this idea to obtain symbolic representations of algebraic $\mathbb{Z}^d$-actions, and gave an example of an algebraic $\mathbb{Z}^2$-action for which the symbolic cover could be pruned to a shift of finite type to obtain a map that is one-to-one almost everywhere, but the proof involved a complicated percolation argument. Even for Heisenberg group actions virtually nothing is known about the existence of good symbolic covers.

Problem 6.6. Find general sufficient conditions on an expansive $f \in \mathbb{Z}\Gamma$ so that the symbolic cover in Proposition 6.1, 1) can be pruned to one that is (a) of finite type or at least sofic, (b) of equal entropy, or (c) one-to-one almost everywhere.

[3] *Russian editor's note*: Vershik's first paper on this subject was "The fibadic expansions of real numbers and adic transformation", Preprint Institut Mittag-Leffler, № 4, 1991, 1–9. In it the idea of arithmetic encoding and symbolic cover for toral automorphisms was realized for the Fibonacci automorphism of the 2-torus (sometimes also called Arnold's cat map). The covering shift is similar to the 'golden mean shift'. The main reason behind arithmetic encoding was to preserve the algebraic structures on the torus and the hyperbolic properties of the automorphism.

7. Entropy of algebraic actions

We give here several equivalent definitions of the (topological) entropy of an algebraic action, sketch some background material on von Neumann algebras, and then describe recent results relating entropy to Fuglede–Kadison determinants.

Let Δ be a countable discrete group. For finite subsets $F, K \subset \Delta$, define $FK = \{\delta\theta\colon \delta \in F,\ \theta \in K\}$. For what follows to make sense, we require that Δ be *amenable*, namely, that there is a sequence $\{F_n\colon n \geqslant 1\}$ of finite subsets of Δ such that for every finite subset K of Δ we have

$$\frac{|F_n \triangle F_n K|}{|F_n|} \to 0 \quad \text{as } n \to \infty.$$

Such a sequence is called a *right-Følner sequence*.

Suppose that α is an algebraic Δ-action on a compact Abelian group X. We assume that there is a translation-invariant metric d on X, and let μ denote normalized Haar measure on X.

As before, we abbreviate the (left) α-action of Δ on X by using $\delta \cdot t$ for $\alpha^\delta(t)$. Then Δ acts on subsets $E \subset X$ by $\delta \cdot E = \{\delta \cdot t\colon t \in E\}$. Although this differs from the traditional action of transformations on subsets using inverse images, this seems better suited to our purposes, since all theα^δ are invertible, and its use is consistent with the action of Δ on functions on X: if χ_E is the indicator function of a set E, then $\delta \cdot \chi_E = \chi_E \circ \delta^{-1} = \chi_{\delta \cdot E}$.

To define topological entropy, we consider open covers $\mathscr{U}$ of X. If $\mathscr{U}_1, \dots, \mathscr{U}_n$ are open covers, define their *span* as $\bigvee_{j=1}^n \mathscr{U}_j = \{U_1 \cap \cdots \cap U_n\colon U_j \in \mathscr{U}_j \text{ for } 1 \leqslant j \leqslant n\}$. If $\mathscr{U}$ is an open cover and F is a finite subset of Δ, then let $\mathscr{U}^F = \bigvee_{\delta \in F} \delta \cdot \mathscr{U}$, where $\delta \cdot \mathscr{U} = \{\delta \cdot U\colon U \in \mathscr{U}\}$. For an open cover $\mathscr{U}$ let $\mathsf{N}(\mathscr{U})$ denote the cardinality of the open subcover with fewest elements, which is finite by compactness. It is easy to check that $\mathsf{N}(\mathscr{U} \vee \mathscr{V}) \leqslant \mathsf{N}(\mathscr{U})\mathsf{N}(\mathscr{V})$. We define the *open cover entropy* of α to be

$$\mathsf{h}_{\mathrm{cov}}(\alpha) = \sup_{\mathscr{U}} \limsup_{n \to \infty} \left(\frac{1}{|F_n|} \log \mathsf{N}(\mathscr{U}^{F_n}) \right), \tag{7.1}$$

where the supremum is taken over all open covers of X.

We recall the elementary fact that if $\{a_n\colon n \geqslant 1\}$ is a sequence of non-negative real numbers with $a_{m+n} \leqslant a_m + a_n$, then a_n/n converges to a limit as $n \to \infty$, and this limit equals $\inf_{1 \leqslant n < \infty} a_n/n$. Hence, for $\Delta = \mathbb{Z}$ and $F_n = \{0, 1, \dots, n-1\}$, it follows that the lim sup in (7.1) is actually a limit. There is a general version of this argument valid for arbitrary amenable groups, due to Lindenstrauss and Weiss.

Proposition 7.1 ([46], Theorem 6.1). *Suppose that $\phi(F)$ is a real-valued function defined for all non-empty finite subsets F of Δ and satisfying the following conditions*:

1) $0 \leqslant \phi(F) < \infty$;
2) *if $F' \subseteq F$, then $\phi(F') \leqslant \phi(F)$*;
3) $\phi(\delta F) = \phi(F)$ *for all $\delta \in \Delta$*;
4) $\phi(F \cup F') \leqslant \phi(F) + \phi(F')$ *if $F \cap F' = \varnothing$.*

Then for every right-Følner sequence $\{F_n\}$ the numbers $\phi(F_n)/|F_n|$ converge to a finite limit, and this limit is independent of the choice of right-Følner sequence.

Roughly speaking, this fact is proved by showing that if K is a large finite subset of Δ and ε is small, then any F with $|F \triangle FK|/|F| < \varepsilon$ can be almost exactly tiled by left translates of F_ns of various sizes. Then the subadditivity and translation-invariance of ϕ show that $\phi(F)/|F|$ is bounded above, within a small error, by $\phi(F_n)/|F_n|$ for sufficiently large n.

Fix an open cover $\mathscr{U}$ of X, and put

$$\phi(F) = \log \mathsf{N}(\mathscr{U}^F)$$

for every non-empty finite subset F of Δ. Since each α^δ is a homeomorphism of X, it follows that

$$\phi(\delta F) = \log \mathsf{N}\Big(\bigvee_{\theta\in F} (\delta\theta)\cdot\mathscr{U} \Big) = \log \mathsf{N}\Big(\delta\cdot\Big(\bigvee_{\theta\in F} \theta\cdot\mathscr{U} \Big)\Big) = \log \mathsf{N}\Big(\bigvee_{\theta\in F} \theta\cdot\mathscr{U} \Big) = \phi(F).$$

Conditions 1), 2), and 4) in Proposition 7.1 are trivially satisfied for this ϕ. Hence, for every open cover $\mathscr{U}$, the lim sup in (7.1) is a limit, and this limit does not depend on the choice of right Følner sequence $\{F_n\}$.

The open cover definition of topological entropy is due to Adler, Konheim, and McAndrew [1]. R. Bowen [8] introduced equivalent definitions that are better suited for many purposes, and we now describe them.

If F is a finite subset of Δ and $\varepsilon > 0$, a subset $E \subset X$ is called (F,ε)-*spanning* if for every $t \in X$ there is a $u \in E$ such that $d(\delta^{-1}\cdot t, \delta^{-1}\cdot u) < \varepsilon$ for every $\delta \in F$. Dually, a set $E \subset X$ is called (F,ε)-*separated* if for distinct elements $t, u \in E$ there is a $\delta \in F$ such that $d(\delta^{-1}\cdot t, \delta^{-1}\cdot u) \geqslant \varepsilon$. Let $r_F(\varepsilon)$ denote the smallest cardinality of any (F,ε)-spanning set, and let $s_F(F,\varepsilon)$ be the largest cardinality of any (F,ε)-separated set. Put

$$\mathsf{h}_{\mathrm{span}}(\alpha) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{|F_n|} \log r_{F_n}(\varepsilon) \quad \text{and} \quad \mathsf{h}_{\mathrm{sep}}(\alpha) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{|F_n|} \log s_{F_n}(\varepsilon).$$

If $\mathscr{U}_\varepsilon$ denotes the open cover of X by ε-balls, then the elementary inequalities

$$\mathsf{N}(\mathscr{U}_\varepsilon^F) \leqslant r_F(\varepsilon) \leqslant s_F(\varepsilon) \leqslant \mathsf{N}(\mathscr{U}_{\varepsilon/2}^F)$$

show that $\mathsf{h}_{\mathrm{cov}}(\alpha) = \mathsf{h}_{\mathrm{span}}(\alpha) = \mathsf{h}_{\mathrm{sep}}(\alpha)$, and so all three quantities are independent of the choice of a right Følner sequence.

One more variant of the entropy definition, using volume decrease, is also useful. Let $B_\varepsilon = \{t \in X\colon d(t,0) < \varepsilon\}$, and for finite $F \subset \Delta$ put $B_\varepsilon^F = \bigcap_{\delta\in F} \delta\cdot B_\varepsilon$. Define

$$\mathsf{h}_{\mathrm{vol}}(\alpha) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{|F_n|} \log \mu(B_\varepsilon^{F_n}).$$

If a subset E is (F,ε)-spanning, then $X = \bigcup_{t\in E}(t + B_\varepsilon^F)$, so that $1 \leqslant |E|\mu(B_\varepsilon^F)$, and hence $\mathsf{h}_{\mathrm{vol}}(\alpha) \leqslant \mathsf{h}_{\mathrm{span}}(\alpha)$. If E is (F,ε)-separated, then the sets $\{t + B_{\varepsilon/2}^F)\colon t \in E\}$ are disjoint, so that $|E|\mu(B_{\varepsilon/2}^F) \leqslant 1$, proving that $\mathsf{h}_{\mathrm{sep}}(\alpha) \leqslant \mathsf{h}_{\mathrm{vol}}(\alpha)$. Thus, all these notions of entropy coincide, and we let $\mathsf{h}(\alpha)$ denote their common value. We remark that these are also equal to the measure-theoretic entropy of α with respect to Haar

measure, but we will not be using this fact. Deninger's paper [15] has complete proofs of these facts, and also of the fact that each lim sup in these definitions can be replaced by lim inf without affecting the results.

If $\Phi \in \mathrm{aut}(\Delta)$ and $\{F_n\}$ is a right Følner sequence, then clearly so is $\{\Phi(F_n)\}$. It follows that $\mathsf{h}(\alpha_{\Phi f}) = \mathsf{h}(\alpha_f)$, that is, the entropy is invariant under a change of variables.

Suppose that α is an algebraic Δ-action on X, and that Y is a closed Δ-invariant subgroup of X. Let α_Y denote the restriction of α to Y, and let $\alpha_{X/Y}$ be the quotient action on X/Y. An important property of entropy is that it adds over the exact sequence $0 \to Y \to X \to X/Y \to 0$.

Theorem 7.2 (The Addition Formula, [38], Corollary 6.3). *Let Δ be an amenable group, let α be an algebraic Δ-action on X, and suppose that Y is a closed, Δ-invariant subgroup of X. Then $\mathsf{h}(\alpha) = \mathsf{h}(\alpha_Y) + \mathsf{h}(\alpha_{X/Y})$.*

The Addition Formula has a long history.[4] The basic approach is to take a Borel cross-section to the quotient map $X \to X/Y$, and regard α as a skew product with base action $\alpha_{X/Y}$ and fiber actions that are affine maps of Y with the same automorphism part α_Y but with different translations. The idea is then to show that the translation parts of these affine maps, being isometries, do not affect the entropy. The case $\Delta = \mathbb{Z}$ was proved by R. Bowen [8], and the case $\Delta = \mathbb{Z}^d$ is handled in [44], Appendix B, using arguments due originally to Thomas [66]. Fiber entropy for amenable actions was dealt with in [68]. There is a serious difficulty in generalizing these ideas to non-commutative groups Δ, namely, the lack of a scaling argument used to eliminate a universal constant due to overlaps of open sets in a cover. However, machinery developed by Ollagnier [52] handles this issue, and this was used by Li to give the most general result cited above.

When $\Delta = \mathbb{Z}^d$ there are explicit formulae for the entropy. First consider the case $\Delta = \mathbb{Z}$. Without loss of generality, we can assume that $f(u) \in \mathbb{Z}[u^{\pm}]$ has the form $f(u) = c_n u^n + \cdots + c_1 u + c_0$ with $c_n c_0 \neq 0$. Factor $f(u)$ over $\mathbb{C}$ as $f(u) = c_n \prod_{j=1}^{n} (u - \lambda_j)$. Then Yuzvinskii [70], [71] showed that

$$\mathsf{h}(\alpha_f) = \log |c_n| + \sum_{j=1}^{n} \log^+ |\lambda_j| = \mathsf{m}(f). \tag{7.2}$$

An interpretation of (7.2) from [45] shows that the term $\sum_{j=1}^{n} \log^+ |\lambda_j|$ is due to geometric expansion, while the term $\log |c_n|$ is due to p-adic expansions for those primes p dividing c_n, an adelic viewpoint that has been useful in other contexts as well.

Mahler measure is defined for polynomials $f \in \mathbb{Z}[u_1^{\pm}, \ldots, u_d^{\pm 1}] = R_d$ by the formulae

$$\mathsf{m}(f) = \int_{\mathbb{S}^d} \log |f| = \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i s_1}, \ldots, e^{2\pi i s_d})| \, ds_1 \cdots ds_d,$$

[4] *Russian editor's note*: This formula is a generalization of the classical Abramov–Rohlin formula for the entropy of a skew product (see Л. М. Абрамов, В. А. Рохлин, "Энтропия косого произведения", Вестн. ЛГУ, 1962, № 7, 5–13 [L. M. Abramov and V. A. Rohlin, "Entropy of a skew product of mappings with invariant measure", Vestnik Leningrad. Univ., 1962, no. 7, 5–13]).

and $\mathsf{M}(f) = \exp\bigl(\mathsf{m}(f)\bigr)$ [50]. One of the main results in [44] is that for non-zero $f \in R_d$ we have $\mathsf{h}(\alpha_f) = \mathsf{m}(f)$. With this information, the entropy for arbitrary algebraic $\mathbb{Z}^d$-actions can easily be found. Let $\mathfrak{a}$ be an ideal in R_d that is not principal. A simple argument ([44], Theorem 4.2) shows that $\mathsf{h}(\alpha_{R_d/\mathfrak{a}}) = 0$. Any finitely generated R_d-module M has a prime filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r$ with $M_j/M_{j-1} \cong R_d/\mathfrak{p}_j$, where the $\mathfrak{p}_j$ are prime ideals in R_d. Then the Addition Formula shows that $\mathsf{h}(\alpha_M) = \mathsf{h}(\alpha_{R_d/\mathfrak{p}_1}) + \cdots + \mathsf{h}(\alpha_{R_d/\mathfrak{p}_r})$, and each summand can be computed from our preceding remarks.

To conclude this discussion of the $\Delta = \mathbb{Z}^d$ case, we point out that there is a complete characterization of principal actions with zero entropy. Recall the definition in § 4 of a generalized cyclotomic polynomial.

Proposition 7.3 [9], [64]. *Let $f \in \mathbb{Z}[u_1^{\pm}, \dots, u_d^{\pm 1}]$. Then $\mathsf{h}(\alpha_f) = 0$ if and only if either f or $-f$ is a product of generalized cyclotomic polynomials times a monomial.*

This result was originally proved by Boyd [9] using deep results of Schinzel, but was later given a simpler and more geometric proof by Smyth [64].

Turning now to non-commutative groups Δ, we sketch some background material on related von Neumann algebras. The functional analysis used can be found, for example, in [13], Chaps. VII and VIII. Let

$$\ell^2(\Delta, \mathbb{C}) = \left\{ w = \sum_{\delta \in \Delta} w_\delta \delta \colon w_\delta \in \mathbb{C} \text{ and } \|w\|_2^2 := \sum_{\delta \in \Delta} |w_\delta|^2 < \infty \right\},$$

which with the standard inner product $\bigl\langle \sum_\delta w_\delta \delta, \sum_\delta v_\delta \delta \bigr\rangle = \sum_\delta w_\delta \overline{v_\delta}$ is a complex Hilbert space. As in the case of $\ell^1(\Delta, \mathbb{C})$, there is a left-action of Δ on $\ell^2(\Delta, \mathbb{C})$ given by $\theta \cdot \bigl(\sum_\delta w_\delta \delta\bigr) = \sum_\delta w_\delta \, \theta\delta$.

The *group von Neumann algebra* $\mathscr{N}\Delta$ of Δ consists of all those bounded linear operators $T \in \mathscr{B}\bigl(\ell^2(\Delta, \mathbb{C})\bigr)$ commuting with the left Δ-action, that is, such that $T(\delta \cdot w) = \delta \cdot T(w)$. There is a natural inclusion of $\mathbb{C}\Delta$ in $\mathscr{N}\Delta$ given by $f \to \rho_f$, where, as before, $\rho_f(w) = w \cdot f^*$. In addition, there is a faithful normalized trace function $\mathrm{tr}_{\mathscr{N}\Delta} \colon \mathscr{N}\Delta \to \mathbb{C}$ given by $\mathrm{tr}_{\mathscr{N}\Delta}(T) = \langle T(1_\Delta), 1_\Delta \rangle$. This means that $\mathrm{tr}_{\mathscr{N}\Delta}$ is linear, $\mathrm{tr}_{\mathscr{N}\Delta}(1) = 1$, $\mathrm{tr}_{\mathscr{N}\Delta}(TT^*) > 0$ for every $T \neq 0$, and $\mathrm{tr}_{\mathscr{N}\Delta}(ST) = \mathrm{tr}_{\mathscr{N}\Delta}(TS)$.

Using this trace, Fuglede and Kadison [21] defined a determinant function on $\mathscr{N}\Delta$ as follows. Let $T \in \mathscr{N}\Delta$. Then $TT^* \geqslant 0$, so the spectral measure ν of the self-adjoint operator TT^* is supported on $[0, \infty)$: in fact, on $[0, \|TT^*\|]$. Using the functional calculus for $\mathscr{B}\bigl(\ell^2(\Delta, \mathbb{C})\bigr)$, we can form the operator $\log(TT^*) = \int_{0^+}^{\infty} \log t \, d\nu(t)$, where the lower limit 0^+ of the integration indicates that we ignore any point mass at 0 that ν may have. We then define

$$\det_{\mathscr{N}\Delta} T = \exp\left[\frac{1}{2} \mathrm{tr}_{\mathscr{N}\Delta}\bigl(\log(TT^*)\bigr)\right].$$

This Fuglede–Kadison determinant has the following very useful properties (see [47], § 3.2 for details):

(a) $\det_{\mathscr{N}\Delta} T^* = \det_{\mathscr{N}\Delta} T$;

(b) if $T > 0$ in $\mathscr{N}\Delta$, then $\det_{\mathscr{N}\Delta} T = \exp(\mathrm{tr}_{\mathscr{N}\Delta}(\log T))$;

(c) if $0 \leqslant S \leqslant T$ in $\mathscr{N}\Delta$, then $\det_{\mathscr{N}\Delta} S \leqslant \det_{\mathscr{N}\Delta} T$;
(d) $\det_{\mathscr{N}\Delta} ST = (\det_{\mathscr{N}\Delta} S)(\det_{\mathscr{N}\Delta} T)$.

We remark that the multiplicativity of $\det_{\mathscr{N}\Delta}$ is not obvious, essentially being a consequence of the Campbell–Baker–Hausdorff formula and the vanishing of the trace on commutators, although for technical reasons a complex variables approach is more efficient.

Example 7.4. Let $\Delta = \mathbb{Z}^d$, and write elements of $\mathbb{Z}^d$ as $\mathbf{n} = (n_1, \dots, n_d)$. The Fourier transform identifies $\ell^2(\mathbb{Z}^d, \mathbb{C})$ with $L^2(\mathbb{T}^d, \mathbb{C})$, with $\mathbf{n}$ being identified with the function $\chi_{\mathbf{n}}$, where $\chi_{\mathbf{n}}(\mathbf{s}) = \exp\bigl[2\pi i(\mathbf{n} \cdot \mathbf{s})\bigr]$ for $\mathbf{s} \in \mathbb{T}^d$. Any bounded linear operator T on $\ell^2(\mathbb{Z}^d, \mathbb{C})$ commuting with the $\mathbb{Z}^d$-action must have the form of convolution with some element $v \in \ell^2(\mathbb{Z}^d, \mathbb{C})$. Hence, the Fourier transform V of v must give a bounded linear operator on $L^2(\mathbb{T}^d)$ via pointwise multiplication, and this forces $V \in L^\infty(\mathbb{T}^d, \mathbb{C})$. Conversely, every $V \in L^\infty(\mathbb{T}^d, \mathbb{C})$ corresponds to an element of $\mathscr{N}\mathbb{Z}^d$. This identifies $\mathscr{N}\mathbb{Z}^d$ with $L^\infty(\mathbb{T}^d, \mathbb{C})$. Under this identification,

$$\operatorname{tr}_{\mathscr{N}\mathbb{Z}^d}(V) = \int_{\mathbb{T}^d} V(\mathbf{s})\, d\mathbf{s} \quad \text{and} \quad \det_{\mathscr{N}\mathbb{Z}^d} V = \exp\left[\int_{\mathbb{T}^d} \log|V(\mathbf{s})|\, d\mathbf{s}\right].$$

For $f \in \mathbb{Z}\mathbb{Z}^d$, we observed that $\rho_f \in \mathscr{N}\mathbb{Z}^d$, and this corresponds to multiplication by $f(e^{-2\pi i s_1}, \dots, e^{-2\pi i s_d})$. Hence, in this case $\det_{\mathscr{N}\mathbb{Z}^d} f = \mathsf{M}(f)$, and so $\log\det_{\mathscr{N}\mathbb{Z}^d} f = \mathsf{m}(f) = \mathsf{h}(\alpha_f)$.

Indeed, it was the equality of Mahler measure, entropy, and the Fuglede–Kadison determinant in Lück's book [47], Example 3.13 that originally inspired Deninger to investigate whether this phenomenon extended to more general groups. He was able to show in [15] that under some conditions it did. Further work [17], [38] extended the generality, culminating in the comprehensive results of Li and Thom [40].

To describe the last work, we recall that ρ_f is a bounded linear operator on $\ell^2(\Delta, \mathbb{C})$. More generally, if $F \in \mathbb{Z}\Delta^{k\times l}$, then there exists a bounded linear operator $\rho_F\colon \ell^2(\Delta, \mathbb{C})^k \to \ell^2(\Delta, \mathbb{C})^l$ given by right multiplication by F^*, where $(F^*)_{i,j} = (F_{i,j})^*$. There is an extension of $\det_{\mathscr{N}\Delta}$ to such F (see [40], § 2.1 for details).

Theorem 7.5 ([40], Theorem 1.2). *Let Δ be a countable discrete amenable group, and let $f \in \mathbb{Z}\Delta$. Suppose that $\rho_f\colon \ell^2(\Delta, \mathbb{C}) \to \ell^2(\Delta, \mathbb{C})$ is injective. Then $\mathsf{h}(\alpha_f) = \log\det_{\mathscr{N}\Delta} f$. More generally, if $F \in \mathbb{Z}\Delta^{k\times l}$ and if $\rho_F\colon \ell^2(\Delta, \mathbb{C})^k \to \ell^2(\Delta, \mathbb{C})^l$ is injective, then $\mathsf{h}(\alpha_F) \leqslant \log\det_{\mathscr{N}\Delta} F$. If $k = l$, then $\mathsf{h}(\alpha_F) = \mathsf{h}(\alpha_{F^*}) = \log\det_{\mathscr{N}\Delta} F$.*

In particular, $\mathsf{h}(\alpha_f) = \mathsf{h}(\alpha_{f^*})$. This is a highly non-trivial fact, since there is no direct connection between α_f and α_{f^*}.

The computation, or even estimation, of the values of Fuglede–Kadison determinants is not easy. In the next two sections we will explicitly calculate the entropy for certain principal actions of the Heisenberg group.

As pointed out by Deninger [15], there are examples of lopsided polynomials f for which $\det_{\mathscr{N}\Delta} f$ can be computed by a rapidly converging series.

Example 7.6. Let $f(x, y, z) = 5 - x - x^{-1} - y - y^{-1} \in \mathbb{Z}\Gamma$. We write $f = 5(1 - g)$, where $g = (x + x^{-1} + y + y^{-1})/5$. It is easy to see ([5], Lemma 2.7) that the spectrum

of g, considered as an element in $\mathscr{N}\Gamma$, is contained in $[-4/5, 4/5]$. Hence, we can apply the functional calculus to compute $\log f$ via the power series for $\log(1-t)$, yielding

$$\log f = \log 5 + \log(1-g) = \log 5 - \sum_{n=1}^{\infty} \frac{g^n}{n}\,.$$

Now $\mathrm{tr}_{\mathscr{N}\Gamma}(g^n)$ is the value of the constant term of g^n. If $r_\Gamma(n)$ denotes the number of words over $S = \{x, x^{-1}, y, y^{-1}\}$ of length n whose product is 1, then clearly $\mathrm{tr}_{\mathscr{N}\Gamma}(g^n) = 5^{-n} r_\Gamma(n)$. In any word over S with product 1, the number of occurrences of x and x^{-1} must be equal, and similarly with y and y^{-1}, so that $r_\Gamma(n) = 0$ for n odd. The numbers $r_\Gamma(2n)$ grow rapidly:

$$r_\Gamma(2n) = 4^{2n}\left(\frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)\right)$$

(see [28], although the result stated there is off by a factor of 2). We thank David Wilson for providing us with a short *Mathematica* program that computes $r_\Gamma(2n)$ up to $r_\Gamma(60)$, which is a 33 digit number. Hence,

$$\mathrm{tr}_{\mathscr{N}\Gamma} \log f = \log 5 - \sum_{n=1}^{30} \frac{r_\Gamma(2n)}{(2n)\cdot 5^{2n}} - \sum_{n=31}^{\infty} \frac{r_\Gamma(2n)}{(2n)\cdot 5^{2n}}\,.$$

By the trivial estimate $r_\Gamma(2n) \leqslant 4^{2n}$, the last sum has value less than 10^{-7}. The remaining part therefore gives the value for $\mathsf{h}(\alpha_f) = \mathrm{tr}_{\mathscr{N}\Gamma} \log f \cong 1.514708$, correct to six decimal places.

We remark that $5g = x + x^{-1} + y + y^{-1}$, considered as an operator in $\mathscr{N}\Gamma$, has been studied intensively, and is closely related to Kac's famous Ten Martini Problem. Indeed, the image of $5g$ in the rotation algebra factor $\mathscr{A}_\theta$ of $\mathscr{N}\Gamma$ is Harper's operator H_θ (cf. [5]). Kac conjectured that for every irrational θ the spectrum of H_θ is a Cantor set of zero Lebesgue measure. This conjecture was recently confirmed by deep work of Artur Avila.

Remark 7.7. The calculations in the previous example can be carried out just as well in any group Δ containing two elements x and y. The only change is that the number $r_\Delta(n)$ of words over $S = \{x, x^{-1}, y, y^{-1}\}$ whose product is 1 will be different, depending on Δ. For comparison with the example, we work this out for $\Delta = \mathbb{Z}^2$ with commuting generators x and y, and for $\Delta = \mathrm{F}_2$, the free group with generators x and y.

Recalling that $f = 5(1-g)$, let us define

$$L(f, \Delta) := \log 5 - \sum_{n=1}^{\infty} \frac{\mathrm{tr}\, g^n}{n} = \log 5 - \sum_{n=1}^{\infty} \frac{r_\Delta(n)}{n \cdot 5^n}\,.$$

The preceding Example 7.6 shows that $L(f, \Gamma) \cong 1.514708$.

For $\Delta = \mathbb{Z}^2$, we are computing the entropy for f considered as a polynomial in the commuting variables x and y, and this is given by Mahler measure to be

$$L(f, \mathbb{Z}^2) = \int_0^1 \int_0^1 \log|5 - 2\cos(2\pi s) - 2\cos(2\pi t)|\, ds\, dt \cong 1.507982.$$

Observe that any word in $S \subset \Gamma$ whose product is 1 Abelianizes to the unity in $\mathbb{Z}^2$, and so $r_\Gamma(n) \leqslant r_{\mathbb{Z}^2}(n)$, which is reflected in the inequality $L(f,\Gamma) \geqslant L(f,\mathbb{Z}^2)$.

The case $\Delta = \mathrm{F}_2$ is more interesting. Here any word in the generators x, y whose product is 1 must also give a word in any group Δ containing x and y with product 1, so that $r_{\mathrm{F}_2}(n) \leqslant r_\Delta(n)$ for all groups Δ. Hence, $L(f,\mathrm{F}_2) \geqslant L(f,\Delta)$, so that $L(f,\mathrm{F}_2)$ gives a universal upper bound.

To compute $L(f,\mathrm{F}_2)$, start with the generating function for $r_{\mathrm{F}_2}(2n)$, which by [14], § I. 9 is known to be

$$G(t) = \sum_{n=1}^{\infty} r_{\mathrm{F}_2}(2n)t^n = \frac{3}{1+2\sqrt{1-12t}} = 1 + 4t + 28t^2 + 232t^3 + 2092t^4 + \cdots .$$

Letting

$$H(t) = \int_0^t \frac{G(u)-1}{u}\,du,$$

we get from a calculation with *Mathematica* that

$$L(f,\mathrm{F}_2) = \log 5 - \frac{1}{2}H(5^{-2}) = \log\left[\frac{1}{18}(35+13\sqrt{13})\right] \cong 1.514787.$$

Indeed, $L(f,\mathrm{F}_2) > L(f,\Gamma)$, but the difference appears only in the fifth decimal place.

We remark that L. Bowen [7] has extended a notion of entropy to actions of sofic groups, and in particular free groups (although these do not have Følner sequences). In [7], Example 1.1 he shows that in the previous discussion $L(f,\mathrm{F}_2)$ equals the sofic entropy of the corresponding algebraic F_2-action.

It is somewhat surprising that here $L(f,\mathrm{F}_2)$ is the logarithm of an algebraic number, since in the case $\Delta = \mathbb{Z}^d$ with $d > 1$ this appears not to generally be the case. For example, when $d = 3$ and $g = 1 + u_1 + u_2 + u_3$, Smyth [65] has computed the logarithmic Mahler measure to be $\mathsf{m}(g) = \mathsf{h}(\alpha_g) = \log[7\zeta(3)/2\pi^2]$, where ζ is the Riemann zeta-function.

Next we extend the 'face entropy' inequality described for principal $\mathbb{Z}^d$-actions in [44], Remark 5.5 to principal $\mathbb{Z}\Gamma$-actions. This inequality proves that many such actions have strictly positive entropy.

We start with the basic case.

Proposition 7.8. *Let* $f(x,y,z) = \sum_{r=0}^{D} g_r(x,z)y^r \in \mathbb{Z}\Gamma$ *with* $g_0(x,z) \neq 0$. *Then* $\mathsf{h}(\alpha_f) \geqslant \mathsf{m}(g_0)$.

Proof. By definition the map $\rho_{g_0(x,z)}\colon \mathbb{T}^{\mathbb{Z}^2} \to \mathbb{T}^{\mathbb{Z}^2}$ has kernel X_{g_0}, and is surjective since multiplication by $g_0(x,z)$ is injective on $\mathbb{Z}[x^{\pm},z^{\pm}]$. Define $\Phi\colon \mathbb{T}^{\mathbb{Z}^2} \to \mathbb{T}^{\mathbb{Z}^2}$ by $(\Phi u)_{i,j} = u_{i+j,j}$. Then for every $k \in \mathbb{Z}$ we get that $u \in \ker \rho_{g_0(x,z)}$ if and only if $\Phi^k(u) \in \ker \rho_{g_0(xz^{-k},z)}$.

The algebraic $\mathbb{Z}^2$-action α_{g_0} on X_{g_0} has entropy $\mathsf{m}(g_0)$. Fix $\varepsilon,\delta > 0$. Then for sufficiently large rectangles $Q \subset \mathbb{Z}^2$ there is a (Q,ε)-separated set $\{u_1,\dots,u_N\} \subset X_{g_0}$ with $N \geqslant e^{(\mathsf{m}(g_0)-\delta)|Q|}$. Note that since $d_{\mathbb{T}}$ is translation invariant, for every $u \in \mathbb{T}^{\mathbb{Z}^2}$ the translated set $\{u_1+u,\dots,u_N+u\}$ is also (Q,ε)-separated.

Let

$$t^{(n)} = \sum_{i,j} t^{(n)}_{i,j} x^i z^j \in \mathbb{T}^{\mathbb{Z}^2}$$

be arbitrary points, and put $t = \sum_{n=-\infty}^{\infty} t^{(n)} y^n \in \mathbb{T}^\Gamma$. The condition for t to be in X_f is that $\rho_f(t) = 0$, which in terms of the $t^{(n)}$ becomes

$$\sum_{r=0}^{D} \rho_{g_r(xz^k,z)}(t^{(k+r)}) = 0 \quad \text{for all } k \in \mathbb{Z}. \tag{7.3}$$

Let $L \geqslant 1$. For each $(j_0, j_1, \dots, j_{L-1}) \in \{1, \dots, N\}^L$ we will construct a sequence $\{t^{(0)}_{j_0}, t^{(-1)}_{j_0,j_1}, \dots, t^{(-L+1)}_{j_0,j_1,\dots,j_{L-1}}\}$ in $\mathbb{T}^{\mathbb{Z}^2}$ that will be used to create an ε-separated set in X_f.

Put $t^{(n)} = 0$ for all $n \geqslant 1$, so that (7.3) is trivially satisfied for $k \geqslant 1$. Define $t^{(0)}_{j_0} = u_{j_0}$ for $1 \leqslant j_0 \leqslant N$. Then (7.3) is satisfied at $k = 0$ because $\rho_{g_0(x,z)}(u_{j_0}) = 0$.

Since $\rho_{g_0(xz^{-1},z)}$ is surjective on $\mathbb{T}^{\mathbb{Z}^2}$, for each j_0 there is a $t^{(-1)}_{j_0} \in \mathbb{T}^{\mathbb{Z}^2}$ such that

$$\rho_{g_0(xz^{-1},z)}\big(t^{(-1)}_{j_0}\big) = -\rho_{g_1(xz^{-1},z)}\big(t^{(0)}_{j_0}\big).$$

Let $t^{(-1)}_{j_0,j_1} = t^{(-1)}_{j_0} + \Phi(u_{j_1})$ for $1 \leqslant j_1 \leqslant N$. Since $\rho_{g_0(xz^{-1},z)}\big(\Phi(u_{j_1})\big) = 0$, it follows that (7.3) is satisfied at $k = -1$ for all choices of j_0 and j_1.

Similarly, for each $\{t^{(0)}_{j_0}, t^{(-1)}_{j_0,j_1}\}$ there is a $t^{(-2)}_{j_0,j_1} \in \mathbb{T}^{\mathbb{Z}^2}$ with

$$\rho_{g_0(xz^{-2},z)}\big(t^{(-2)}_{j_0,j_1}\big) = -\rho_{g_1(xz^{-2},z)}\big(t^{(-1)}_{j_0,j_1}\big) - \rho_{g_2(xz^{-2},z)}\big(t^{(0)}_{j_0}\big).$$

Put $t^{(-2)}_{j_0,j_1,j_2} = t^{(-2)}_{j_0,j_1} + \Phi^2(u_{j_2})$ for $1 \leqslant j_2 \leqslant N$. Then (7.3) holds for $k = -2$ and all choices of j_0, j_1, j_2.

Continuing in this way, for every L-tuple $(j_0, j_1, \dots, j_{L-1}) \in \{1, \dots, N\}^L$ we have constructed a sequence $\{t^{(0)}_{j_0}, t^{(1)}_{j_0,j_1}, \dots, t^{(-L+1)}_{j_0,\dots,j_{L-1}}\}$ so that (7.3) is satisfied for $k \geqslant -L+1$. Each choice can be further extended to find $t^{(n)}$ for $n \leqslant -L$ for which the resulting point is in X_f.

Identify $\mathbb{Z}^3$ with Γ via the rule $(i,j,k) \leftrightarrow x^i z^j y^k$, and consider the set

$$\overline{Q} = \bigcup_{k=0}^{L-1} \Phi^k(Q) \times \{-k\} \subset \Gamma.$$

We claim that the N^L points in X_f constructed above are $(\overline{Q}, \varepsilon)$-separated. For at the first index k for which $j_k \neq j'_k$, the points $\Phi(u_{j_k})$ and $\Phi^k(u_{j'_k})$ differ by at least ε at some coordinate of $\Phi^k(Q)$.

Finally, if we choose Q to be very long in the x-direction as compared with the z-direction, and make L small compared with both quantities, then we can make $\overline{Q}$ as right-invariant as we please. Hence, there is a Følner sequence $\{\overline{Q}_m\}$ in Γ with

$$s(\overline{Q}_m, \varepsilon) \geqslant e^{(\mathsf{m}(g_0)-\delta)|\overline{Q}_m|},$$

and thus $\mathsf{h}(\alpha_f) \geqslant \mathsf{m}(g_0)$. $\square$

Recall from § 2 that the *Newton polygon* $\mathscr{N}(f)$ of an $f = \sum_{k,l} f_{kl}(z)x^k y^l \in \mathbb{Z}\Gamma$ is the convex hull in $\mathbb{R}^2$ of those points (k,l) for which $f_{kl}(z) \neq 0$. A *face* of $\mathscr{N}(f)$ is the intersection of $\mathscr{N}(f)$ with a supporting hyperplane, and thus is either a point or a line segment. For each face F of $\mathscr{N}(f)$ let

$$f_F(x,y,z) = \sum_{(k,l)\in F} f_{k,l}(z)x^k y^l.$$

For every face F of $\mathscr{N}(f)$ there is a change of variables followed by multiplication by a monomial transforming f so that F now lies on the x-axis with the rest of $\mathscr{N}(f)$ in the upper half-plane. Since entropy is invariant under such transformations, we can apply Proposition 7.8 to obtain the following face entropy inequality.

Corollary 7.9. *If $f \in \mathbb{Z}\Gamma$ and F is a face of $\mathscr{N}(f)$, then $\mathsf{h}(\alpha_f) \geqslant \mathsf{h}(\alpha_{f_F})$.*

Face entropies are essentially logarithmic Mahler measures of polynomials in commuting variables, and so are easy to compute. Observe that if $\mathsf{h}(\alpha_f) = 0$, then the Corollary shows that $h(\alpha_{f_F}) = 0$ for every face F of $\mathscr{N}(f)$, and then Proposition 7.3 gives a complete characterization of what f_F can be. Indeed, Smyth used the face entropy inequality as the starting point for his proof of Proposition 7.3. However, the algebraic complexity of $\mathbb{Z}\Gamma$ prevents a direct extension of his methods, leaving open a very interesting question.

Problem 7.10. Characterize those $f \in \mathbb{Z}\Gamma$ for which $\mathsf{h}(\alpha_f) = 0$.

We recall that the Pinsker σ-algebra of a measure-preserving action α is the largest σ-algebra on which the entropy of α is zero. An action has *completely positive entropy* if its Pinsker σ-algebra is trivial. An old argument of Rohlin shows that the Pinsker σ-algebra of an algebraic action α on X is invariant under translation by any periodic point. Hence if the periodic points are dense, then the Pinsker σ-algebra is invariant under all translations, and so arises from the quotient map $X \to X/Y$, where Y is a compact α-invariant subgroup. Thus, the restriction of α to its Pinsker σ-algebra is again an algebraic action (see [44], Proposition 6.2), providing one reason for the importance of the above problem. For algebraic $\mathbb{Z}^d$-actions there is an explicit criterion for completely positive entropy in terms of associated prime ideals (see [44], Theorem 6.5), but even in the case of Heisenberg actions no similar criterion is known.

Problem 7.11. Characterize the algebraic Γ-actions with a completely positive entropy.

For algebraic $\mathbb{Z}^d$-actions, completely positive entropy is sufficient to imply that they are isomorphic to Bernoulli shifts [58]. Is the same true for algebraic Γ-actions?

Problem 7.12. If an algebraic Γ-action has completely positive entropy, is it necessarily measurably isomorphic to a Bernoulli Γ-action?

8. Periodic points and entropy

Let Δ be a countable discrete group and α an algebraic Δ-action on X. A point $t \in X$ is *periodic* for α if its Δ-orbit is finite. The stabilizer $\{\delta \in \Delta\colon \delta \cdot t = t\}$

of such a point t has finite index in Δ, and we will need a generous supply of such subgroups. Call Δ *residually finite* if, for every finite subset K of Δ, there is a finite-index subgroup Λ of Δ such that $\Lambda \cap (K \setminus \{1\}) = \varnothing$. Every finite-index subgroup Λ of Δ contains a further finite-index subgroup Λ' of Λ that is normal in Δ, so that residual finiteness can be defined using finite-index normal subgroups. If $\{\Lambda_n\}$ is a sequence of finite-index subgroups of Δ, then we will say that $\Lambda_n \to \infty$ if, for every finite set $K \subset \Delta$, there is an n_K such that $\Lambda_n \cap (K \setminus \{1\}) = \varnothing$ for all $n \geqslant n_K$.

For a finite-index subgroup Λ of Δ, let

$$\mathsf{Fix}_\Lambda(\alpha) := \big\{t \in X \colon \lambda \cdot t = t \text{ for all } \lambda \in \Lambda\big\}.$$

If Λ is normal in Δ, then for $\lambda \in \Lambda$ and $\delta \in \Delta$ we have $\delta\lambda\delta^{-1} = \lambda' \in \Lambda$. Hence, in this case $\mathsf{Fix}_\Lambda(\alpha)$ is Δ-invariant, for if $t \in \mathsf{Fix}_\Lambda(\alpha)$, then $\lambda \cdot (\delta \cdot t) = (\lambda\delta) \cdot t = (\delta\lambda') \cdot t = \delta \cdot (\lambda' \cdot t) = \delta \cdot t$.

We will focus on expansive principal actions. Let Δ be a countable residually finite discrete group, and let $f \in \mathbb{Z}\Delta$ be expansive. Recall the notation and results from Proposition 6.1. For a finite-index subgroup Λ of Δ, let the superscript Λ on a space denote the set of those elements in the space that are fixed by Λ, so, for example,

$$\ell^\infty(\Delta, \mathbb{R})^\Lambda = \big\{w \in \ell^\infty(\Delta, \mathbb{R}) \colon \lambda \cdot t = t \text{ for all } \lambda \in \Lambda\big\}.$$

Proposition 8.1 ([17], Proposition 5.2). *Let Δ be a countable discrete group, and let $f \in \mathbb{Z}\Delta$ be expansive. With the notation of Proposition 6.1, every finite-index subgroup Λ of Δ satisfies*

$$\mathsf{Fix}_\Lambda(\alpha_f) = \pi\big(\ell^\infty(\Delta, \mathbb{Z})^\Lambda\big) \cong \ell^\infty(\Delta, \mathbb{Z})^\Lambda / \rho_f\big(\ell^\infty(\Delta, \mathbb{Z})^\Lambda\big).$$

Proof. Any point $t \in \mathsf{Fix}_\Lambda(\alpha_f)$ can be lifted to a point $\widetilde{t} \in \ell^\infty(\Delta, \mathbb{R})^\Lambda$, and then the proof of Proposition 6.1 yields Λ-invariant points at every step. □

Note that $\ell^\infty(\Delta, \mathbb{Z})^\Lambda$ is a free Abelian group of rank $[\Delta : \Lambda]$, the index of Λ in Δ, and that ρ_f is an injective endomorphism of this group by expansiveness.

Corollary 8.2. *Under the hypotheses in Proposition 8.1,*

$$|\,\mathsf{Fix}_\Lambda(\alpha_f)| = \big|\det(\rho_f|_{\ell^\infty(\Delta,\mathbb{R})^\Lambda})\big|. \tag{8.1}$$

Proof. Observe that ρ_f is an injective linear map on the real vector space $\ell^\infty(\Delta, \mathbb{R})^\Lambda$ of dimension $n = [\Delta : \Lambda]$ and maps the lattice $\ell^\infty(\Delta, \mathbb{Z})^\Lambda$ to itself. If A is any $n \times n$ matrix with integer entries and non-zero determinant, then it is easy to see, for example, from the Smith normal form, that $|\mathbb{Z}^n / A\mathbb{Z}^n| = |\det(A)|$. □

Remark 8.3. Since we will need to complexify some spaces in order to use complex eigenvalues, let us say a word about conventions regarding determinants. If A is an $n \times n$ real matrix, we could regard A as an $n \times n$ complex matrix acting on $\mathbb{C}^n$, or as a $(2n) \times (2n)$ real matrix acting on $\mathbb{R}^n \oplus i\mathbb{R}^n$, and these have different determinants. We will always use the first interpretation.

With Corollary 8.2 we begin to see the connections among periodic points, entropy, and Fuglede–Kadison determinants. For expansive actions, periodic points are separated, and so $|\Lambda|^{-1}\log|\mathsf{Fix}_\Lambda(\alpha_f)|$ should approximate, or at least provide a lower bound for, the entropy $\mathsf{h}(\alpha_f)$. But by (8.1), this is also a finite-dimensional approximation to the logarithm $\log\det_{\mathscr{N}\Delta} f$ of the Fuglede–Kadison determinant of ρ_f. The main technical issue is then to show that both of these approximations converge to the desired limits. For expansive α_f this is relatively easy, but for general α_f there are numerous difficulties to overcome.

We can now deduce two important properties of expansive principal actions.

Proposition 8.4. *Let Δ be a residually finite countable discrete amenable group, and let $f \in \mathbb{Z}\Delta$ be expansive. Then*

1) *the α_f-periodic points are dense in X_f, and*
2) *if $|X_f| > 1$ (that is, if f is not invertible in $\mathbb{Z}\Delta$), then $\mathsf{h}(\alpha_f) > 0$.*

Sketch of proof. 1) Let $t \in X_f$ and find a $u \in \ell^\infty(\Delta,\mathbb{Z})$ with $\pi(u) = \beta(u\cdot w^\Delta) = t$. By Proposition 6.1, 4), there is a finite subset K of Δ such that if $u_K \in \ell^\infty(\Delta,\mathbb{Z})$ denotes the restriction of u to K and 0 elsewhere, then $\pi(u_K)$ is close to t. By enlarging K if necessary, we can assume that $\sum_{\delta\notin K}|w^\Delta_\delta|$ is small. Let Λ be a finite-index subgroup of Δ such that $\lambda K \cap \lambda' K = \varnothing$ for distinct $\lambda,\lambda' \in \Lambda$, and let $t_0 = \sum_{\lambda\in\Lambda}\lambda\cdot\pi(u_K)$. Then $t_0 \in \mathsf{Fix}_\Lambda(\alpha_f)$ and t_0 is close to t.

2) Again choose a finite subset K such that $\sum_{\delta\notin K}|w^\Delta_\delta|$ is small, where we may assume that $w^\Delta_1 = 1$. Find a subgroup Λ with $\lambda K\cap\lambda' K = \varnothing$ for distinct $\lambda,\lambda' \in \Lambda$. If F is any almost invariant Følner set, then $|F\cap\Lambda|$ is about equal to $|F|/[\Delta:\Lambda]$. Then for each choice of $b_\lambda = 0$ or 1 for $\lambda \in F\cap\Lambda$, the points $\sum_{\lambda\in F\cap\Lambda} b_\lambda(\lambda\cdot w^\Delta)$ are $(F,1/2)$-separated. Hence if $\{F_n\}$ is any Følner sequence, then for every $\varepsilon < 1/2$ we have

$$\limsup_{n\to\infty}\frac{1}{|F_n|}\log s(F_n,\varepsilon) \geqslant \limsup_{n\to\infty}\frac{1}{|F_n|}|F_n\cap\Lambda|\log 2 \geqslant \frac{\log 2}{[\Delta:\Lambda]} > 0.$$

For $\Delta = \mathbb{Z}^d$, it turns out that the periodic points for α_M are always dense in X_M for any finitely generated $\mathbb{Z}\mathbb{Z}^d$-module M ([60], Corollary 11.3). The simple example of multiplication by $3/2$ on $\mathbb{Q}$ dualizes to an automorphism of a compact group with no non-zero periodic points, since $(3/2)^n - 1$ is invertible in $\mathbb{Q}$ (see [60], Example 5.6(1)). We do not know the answer to the following.

Problem 8.5. Let Δ be a countable discrete residually finite group, and let M be a finitely generated $\mathbb{Z}\Delta$-module. Must the α_M-periodic points always be dense in X_M?

We now focus on using periodic points to calculate entropy. It is instructive to see how these calculations work in a simple example.

Example 8.6. Let $\Delta = \mathbb{Z}$, and let $f(u) = u^2 - u - 1 \in \mathbb{Z}\Delta = \mathbb{Z}[u^{\pm}]$. Let $\Lambda_n = n\mathbb{Z}$ and $F_n = \{0,1,\dots,n-1\}$. Then $\{F_n\}$ is a Følner sequence in $\mathbb{Z}$ that is also a fundamental domain for Λ_n. Denote by Ω_n the set of all nth roots of unity in $\mathbb{C}$. For $\zeta \in \Omega_n$ let $v_\zeta = \sum_{k\in\mathbb{Z}}\zeta^k u^k \in \ell^\infty(\mathbb{Z},\mathbb{C})^{\Lambda_n}$. Then $\rho_u(v_\zeta) = \zeta v_\zeta$, so the v_ζ form an eigenbasis for the shift opetrator on $\ell^\infty(\mathbb{Z},\mathbb{C})^{\Lambda_n}$. Hence, $\rho_f(v_\zeta) = f(\zeta)v_\zeta$ for each $\zeta \in \Omega_n$.

We can consider the elements of $\ell^\infty(\mathbb{Z},\mathbb{C})^{\Lambda_n}$ as elements in the n-dimensional complex vector space $\ell^\infty(\mathbb{Z}/n\mathbb{Z},\mathbb{C})$. The matrix of ρ_f with respect to the basis $\{1,u,\dots,u^{n-1}\}$ is the circulant matrix

$$C_n(f)=\begin{bmatrix}-1&-1&1&0&\dots&0&0\\0&-1&-1&1&\dots&0&0\\\dots&\dots&\dots&\dots&\ddots&\dots&\dots\\1&0&\dots&\dots&\dots&-1&-1\\-1&1&\dots&\dots&\dots&0&-1\end{bmatrix}. \tag{8.2}$$

We can compute the determinant of $C_n(f)$ using the eigenbasis $\{v_\zeta\colon \zeta\in\Omega_n\}$. Factor $f(u)$ as $(u-\tau)(u-\sigma)$, where $\tau=(1+\sqrt{5}\,)/2$ and $\sigma=-1/\tau$. Then

$$\begin{aligned}|\operatorname{Fix}_{\Lambda_n}(\alpha_f)| &= \bigl|\det\rho_f\big|_{\ell^\infty(\mathbb{Z}/n\mathbb{Z},\mathbb{C})}\bigr| = \prod_{\zeta\in\Omega_n}|f(\zeta)|\\ &= \prod_{\zeta\in\Omega_n}|\tau-\zeta|\,|\sigma-\zeta| = |\tau^n-1|\,|\sigma^n-1|.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n}\log|\operatorname{Fix}_{\Lambda_n}(\alpha_f)| = \log\tau = \mathsf{m}(f).$$

Since α_f is expansive, the periodic points are separated, and so $\log\tau$ is certainly a lower bound for $\mathsf{h}(\alpha_f)$. But it is also easy in this case to see that it is an upper bound, using, for example, approximations by homoclinic points as in the proof of Proposition 8.4.

Note that if $f\in\mathbb{Z}[u^{\pm}]$ had a root $\xi\in\mathbb{S}$, then the factor $|\xi^n-1|$ in the calculation of the determinant would occasionally be very small, which could cause the limit not to exist. This is one manifestation of the difficulties with non-expansive actions.

We turn to the Heisenberg case $\Delta=\Gamma$. For $q,r,s>0$ put $\Lambda_{rq,sq,q}=\langle x^{rq},y^{sq},z^q\rangle$, which is a normal subgroup of Γ of index rsq^3. Let $f\in\mathbb{Z}\Gamma$ be expansive. Recall that $\kappa_0=\kappa_0(f)=1/(3\|f\|_1)$ is an expansive constant for α_f. In particular, if Λ is a finite-index subgroup of Γ, and if $t\neq u\in\operatorname{Fix}_\Lambda(\alpha_f)$, then for any fundamental domain Q of Λ there is a $\gamma\in Q$ such that $d_{\mathbb{T}}(t_\gamma,u_\gamma)\geqslant\kappa_0$.

A bit of notation about the limits we will be taking is convenient. If $\psi(\Lambda_{rq,sq,q})$ is a quantity that depends on $\Lambda_{rq,sq,q}$, then we write $\lim_{q\to\infty}\psi(\Lambda_{rq,sq,q})=c$ to mean that for every $\varepsilon>0$ there is a q_0 such that $|\psi(\Lambda_{rq,sq,q})-c|<\varepsilon$ for every $q\geqslant q_0$ and all sufficiently large r and s.

Theorem 8.7 ([17], Theorem 5.7). *Let $f\in\mathbb{Z}\Gamma$ be expansive, and define $\Lambda_{rq,sq,q}$ as above. Then*

$$\lim_{q\to\infty}\frac{1}{[\Gamma:\Lambda_{rq,sq,q}]}\log|\operatorname{Fix}_{\Lambda_{rq,sq,q}}(\alpha_f)| = \mathsf{h}(\alpha_f).$$

Proof. In the light of Example 8.6, it is tempting to use for $\Lambda_{rq,sq,q}$ a fundamental domain of the form $Q=\{x^ky^lz^m\colon 0\leqslant k<rq,\,0\leqslant l<sq,\,0\leqslant m<q\}$, but such a Q is far from being a right-Følner set, since right multiplication by x will drastically shear Q in the z-direction.

The method used in [17], Theorem 5.7 is to decompose Q into pieces, each of which is thin in the y-direction, and translate these pieces to different locations in Γ. The union of these translates will still be a fundamental domain, but now it will also be Følner, and so can be used for entropy calculations. This method depends in general on a result of Weiss [69], and ultimately goes back to the ε-quasi-tiling machinery of Ornstein and Weiss [53]. In our case, we can give a simple description of this decomposition.

Choose integers $a(q)$ such that $a(q) \to \infty$ but $a(q)/q \to 0$ as $q \to \infty$, and consider the set

$$F_{rq,L,q} = \{x^k y^l z^m : 0 \leqslant k < rq,\ L \leqslant l < L + a(q),\ 0 \leqslant m < q\}.$$

It is easy to verify that since $a(q)$ is small compared with q, right multiplication by a fixed $\gamma \in \Gamma$ creates only small distortions, and so for every $\gamma \in \Gamma$ we have

$$\lim_{q\to\infty} \frac{|F_{rq,L,q} \triangle F_{rq,L,q}\gamma|}{|F_{rq,L,q}|} = 0. \tag{8.3}$$

Define

$$Q_{rq,sq,q} = \bigcup_{j=0}^{[rq/a(q)]-1} x^{rqj} F_{rq,a(q)j,q},$$

where we make the obvious modification in the last set in the union if $a(q)$ does not evenly divide rq. Then $Q_{rq,sq,q}$ is also a fundamental domain for $\Lambda_{rq,sq,q}$, but now it is also a Følner sequence as $q \to \infty$ by (8.3). By the separation property of the periodic points, for all $\varepsilon < \kappa_0$ we have $|\operatorname{Fix}_{\Lambda_{rq,sq,q}}(\alpha_f)| \leqslant s(Q_{rq,sq,q}, \varepsilon)$. Since $\{Q_{rq,sq,q}\}$ is a Følner sequence,

$$\limsup_{q\to\infty} \frac{1}{|Q_{rq,sq,q}|} \log |\operatorname{Fix}_{Q_{rq,sq,q}}(\alpha)| \leqslant \mathsf{h}(\alpha_f).$$

For the reverse inequality, let $\delta > 0$ and let $\varepsilon < \delta/3$. Choose a finite set $E \subset \Gamma$ such that $\sum_{\gamma\notin E} |w_\gamma^\Delta| < \dfrac{\varepsilon}{\|f\|_1}$. The sets

$$P_{rq,sq,q} = \bigcap_{\gamma\in E} Q_{rq,sq,q}\gamma$$

also form a Følner sequence, and $|P_{rq,sq,q}|/|Q_{rq,sq,q}| \to 1$ as $q \to \infty$.

Fix q, r, and s for the moment, and choose a $(P_{rq,sq,q}, \delta)$-separated set S of maximal cardinality. For every $t \in S$, let $\widetilde{t} \in \ell^\infty(\Gamma, \mathbb{R})$ be its lift, with $\|\widetilde{t}\|_\infty \leqslant 1$ and $\beta(\widetilde{t}) = t$. Write $v(t) \in \ell^\infty(\Gamma, \mathbb{Z})^{\Lambda_{rq,sq,q}}$ for the unique point with $v(t)_\gamma = (\rho_f(\widetilde{t}))_\gamma$ for all $\gamma \in Q_{rq,sq,q}$. Our choice of E implies that the points in $\{\pi(v(t)) : t \in S\} \subset \operatorname{Fix}_{\Lambda_{rq,sq,q}}(\alpha_f)$ are $(P_{rq,sq,q}, \delta/3)$-separated, hence distinct. Thus, $|S| \leqslant |\operatorname{Fix}_{\Lambda_{rq,sq,q}}(\alpha_f)|$. Since $\{P_{rq,sq,q}\}$ is Følner and $|P_{rq,sq,q}|/|Q_{rq,sq,q}| \to 1$ as $q \to \infty$, we see that

$$\mathsf{h}(\alpha_f) = \liminf_{q\to\infty} \frac{1}{|P_{rq,sq,q}|} \log s(P_{rq,sq,q}, \delta) \leqslant \liminf_{q\to\infty} \frac{1}{|Q_{rq,sq,q}|} \log |\operatorname{Fix}_{\Lambda_{rq,sq,q}}(\alpha_f)|,$$

completing the proof. □

We apply this result, combined with Corollary 8.2, to compute the entropy for the expansive principal Γ-actions α_f. In the above notation let $V_{rq,sq,q} = \ell^\infty(\Gamma,\mathbb{C})^{\Lambda_{rq,sq,q}}$, so that

$$|\operatorname{Fix}_{\Lambda_{rq,sq,q}}(\alpha_f)| = \big|\det(\rho_f|_{V_{rq,sq,q}})\big|. \tag{8.4}$$

We will compute this determinant by decomposing $V_{rq,sq,q}$ into ρ_f-invariant subspaces, each having dimension 1 or q. To do this, for each $(\xi,\eta,\zeta)\in\mathbb{S}^3$ let

$$v_{\xi,\eta,\zeta} = \sum_{k,l,m=-\infty}^{\infty} \xi^k\eta^l\zeta^m\, x^k y^l z^m \in \ell^\infty(\Gamma,\mathbb{C}).$$

Observe that $\rho_y(v_{\xi,\eta,\zeta}) = v_{\xi,\eta,\zeta}\cdot y^{-1} = \eta\, v_{\xi,\eta,\zeta}$, and similarly $\rho_z(v_{\xi,\eta,\zeta}) = \zeta\, v_{\xi,\eta,\zeta}$, so that $v_{\xi,\eta,\zeta}$ is a common eigenvector for ρ_y and ρ_z. However,

$$\begin{aligned}\rho_x(v_{\xi,\eta,\zeta}) &= \sum_{k,l,m} \xi^k\eta^l\zeta^m\, x^k y^l z^m\cdot x^{-1} = \sum_{k,l,m} \xi^k\eta^l\zeta^m\, x^{k-1} y^l z^{m-l}\\ &= \sum_{k,l,m} \xi^{k+1}(\eta\zeta)^l\zeta^m\, x^k y^l z^m = \xi\, v_{\xi,\eta\zeta,\zeta}.\end{aligned}$$

Let $\Omega_q = \{\zeta\in\mathbb{S}\colon \zeta^q = 1\}$ and $\Omega_q' = \Omega_q\setminus\{1\}$. For arbitrary $(\xi,\eta)\in\mathbb{S}^2$ let

$$W_\zeta(\xi,\eta) = \begin{cases} \bigoplus_{j=0}^{q-1} \mathbb{C} v_{\xi,\eta\zeta^j,\zeta} & \text{if } \zeta\in\Omega_q',\\ \mathbb{C} v_{\xi,\eta,1} & \text{if } \zeta = 1.\end{cases}$$

By the above, for every $(\xi,\eta)\in\mathbb{S}^2$ and $\zeta\in\Omega_q$, the subspace $W_\zeta(\xi,\eta)$ is invariant under the right action of Γ.

Now let $f\in\mathbb{Z}\Gamma$, and assume that f is expansive. Adjusting f by a power of x if necessary, we may assume that f has the form $f(x,y,z) = \sum_{j=0}^{D} x^j g_j(y,z)$, where each $g_j(y,z)$ is in $\mathbb{Z}[y^\pm,z^\pm]$, and $g_0(y,z)\neq 0$ and $g_D(y,z)\neq 0$. The action of ρ_f on $v_{\xi,\eta,\zeta}$ is then given by

$$\rho_f(v_{\xi,\eta,\zeta}) = \sum_{j=0}^{D} \rho_{x^j g_j(y,z)}(v_{\xi,\eta,\zeta}) = \sum_{j=0}^{D} \xi^j g_j(\eta,\zeta) v_{\xi,\eta\zeta^j,\zeta}.$$

If $\zeta = 1$, then $\rho_f(v_{\xi,\eta,1}) = f(\xi,\eta,1)v_{\xi,\eta,1}$, so is given by the 1×1 matrix $A_{1,f}(\xi,\eta) = [f(\xi,\eta,1)]$. If $\zeta\in\Omega_q'$, then the matrix of ρ_f on $W_\zeta(\xi,\eta)$ takes the following $q\times q$ circulant-like form, where for notational convenience we assume that $q > D$:

$$A_{\zeta,f}(\xi,\eta) = \begin{bmatrix} g_0(\eta,\zeta) & g_1(\eta,\zeta)\xi & \dots & g_D(\eta,\zeta)\xi^D & \dots & 0\\ 0 & g_0(\eta\zeta,\zeta) & g_1(\eta\zeta,\zeta)\xi & g_2(\eta\zeta,\zeta)\xi^2 & \dots & 0\\ 0 & 0 & g_0(\eta\zeta^2,\zeta) & g_1(\eta\zeta^2,\zeta)\xi & \dots & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ g_1(\eta\zeta^{q-1},\zeta)\xi & g_2(\eta\zeta^{q-1},\zeta)\xi^2 & \dots & 0 & \dots & g_0(\eta\zeta^{q-1},\zeta)\end{bmatrix}.$$

By our expansiveness assumption ρ_f is injective, and each subspace $W_\zeta(\xi,\eta)$ is ρ_f-invariant, hence $\det A_{\zeta,f}(\xi,\eta) \neq 0$ for all $(\xi,\eta) \in \mathbb{S}^2$ and all $\zeta \in \Omega_q$.

Now suppose that $r, s \geqslant 0$. For convenience we will assume that both r and s are primes distinct from q. Then $\{v_{\xi,\eta,\zeta}\colon (\xi,\eta,\zeta) \in \Omega_{rq} \times \Omega_{sq} \times \Omega_q\}$ is a basis for $V_{rq,sq,q}$. Note that if $\zeta \in \Omega'_q$, then $W_\zeta(\xi,\eta) = W_\zeta(\xi,\eta\zeta^j)$ for $0 \leqslant j < q$. Since $\Omega_{sq} \cong \Omega_s \times \Omega_q$ by the relative primeness of s and q, we can parameterize the spaces $W_\zeta(\xi,\eta)$ by the (new) parameter $\eta \in \Omega_s$. This gives the ρ_f-invariant decomposition

$$V_{rq,sq,q} = \bigoplus \{W_1(\xi,\eta)\colon (\xi,\eta) \in \Omega_{rq} \times \Omega_{sq}\} \oplus \bigoplus \{W_\zeta(\xi,\eta)\colon (\xi,\eta,\zeta) \in \Omega_{rq} \times \Omega_s \times \Omega'_q\}. \tag{8.5}$$

We now evaluate the limit of

$$\frac{1}{rsq^3} \log \left| \det\left(\rho_f|_{V_{rq,sq,q}}\right)\right| \tag{8.6}$$

as $r, s \to \infty$ using the decomposition (8.5).

On each of the 1-dimensional spaces $W_1(\xi,\eta)$ in (8.5) ρ_f acts as multiplication by $f(\xi,\eta,1)$. Hence, the contribution to (8.6) of the first large summand in (8.5) is

$$\frac{1}{rsq^3} \sum_{(\xi,\eta) \in \Omega_{rq} \times \Omega_{sq}} \log|f(\xi,\eta,1)|. \tag{8.7}$$

By expansiveness, $f(\xi,\eta,1)$ never vanishes for $(\xi,\eta) \in \mathbb{S}^2$, so that $\log|f(\xi,\eta,1)|$ is continuous on $\mathbb{S}^2$. By convergence of the Riemann sums to the integral, as $r, s \to \infty$ we have

$$\frac{1}{rsq^2} \sum_{(\xi,\eta) \in \Omega_{rq} \times \Omega_{sq}} \log|f(\xi,\eta,1)| \to \iint_{\mathbb{S}^2} \log|f(\xi,\eta,1)|\, d\xi\, d\eta.$$

The additional factor of q in the denominator of (8.7) shows that it converges to 0 as $r, s \to \infty$.

For the spaces $W_\zeta(\xi,\eta)$ with $\zeta \in \Omega'_q$, the expansiveness assumption shows that $\det A_{\zeta,f}(\xi,\eta)$ never vanishes for $(\xi,\eta) \in \mathbb{S}^2$, so again $\log|\det A_{\zeta,f}(\xi,\eta)|$ is continuous on $\mathbb{S}^2$. Hence, for $\zeta \in \Omega'_q$ we have

$$\frac{1}{|\Omega_{rq} \times \Omega_{sq}|} \sum_{(\xi,\eta) \in \Omega_{rq} \times \Omega_s} \log\left|\det A_{\zeta,f}(\xi,\eta)\right| \to \iint_{\mathbb{S}^2} \log\left|\det A_{\zeta,f}(\xi,\eta)\right| d\xi\, d\eta$$

as $r, s \to \infty$. Adding these up over $\zeta \in \Omega'_q$, and observing that $|\Omega_s| = |\Omega_{qs}|/q$, we have shown the following.

Theorem 8.8. *Let $f(x,y,z) = \sum_{j=0}^{D} x^j g_j(y,z) \in \mathbb{Z}\Gamma$ be expansive. Then*

$$\mathsf{h}(\alpha_f) = \lim_{\substack{q \to \infty \\ q \text{ a prime}}} \frac{1}{q^2} \sum_{\zeta \in \Omega_q} \iint_{\mathbb{S}^2} \log\left|\det A_{\zeta,f}(\xi,\eta)\right| d\xi\, d\eta, \tag{8.8}$$

where the matrices $A_{\zeta,f}(\xi,\eta)$ are as given above.

At first glance the denominator q^2 in (8.8) seems puzzling. The explanation is that one factor q comes from averaging over the qth roots of unity, and the second q comes from the size of the matrices $A_{\zeta,f}(\xi,\eta)$. From the point of view of von Neumann algebras, we should really be using the 'normalized determinant' $|\det A_{\zeta,f}(\xi,\eta)|^{1/q}$ corresponding to the normalized trace on $\mathbb{C}^n$, and then the second q would not appear.

For expansive polynomials in $\mathbb{Z}\Gamma$ that are linear in x the entropy formula in the preceding theorem can be considerably simplified.

Theorem 8.9. *Let $f(x,y,z) = g(y,z) + xh(y,z) \in \mathbb{Z}\Gamma$, where $g(y,z)$ and $h(y,z)$ are Laurent polynomials in $\mathbb{Z}[y^{\pm}, z^{\pm}]$. If α_f is expansive, then*

$$\mathsf{h}(\alpha_f) = \int_{\mathbb{S}} \max\bigl\{\mathsf{m}\bigl(g(\,\cdot\,,\zeta)\bigr), \mathsf{m}\bigl(h(\,\cdot\,,\zeta)\bigr)\bigr\}\, d\zeta, \tag{8.9}$$

where $\mathsf{m}\bigl(g(\,\cdot\,,\zeta)\bigr) = \int_{\mathbb{S}} \log|g(\eta,\zeta)|\, d\eta$ and similarly for h.

Corollary 8.10. *Let $f(x,y,z) = g(y,z)+xh(y,z)$, and assume that neither g nor h vanish anywhere on $\mathbb{S}^2$. If α_f is expansive, then $\mathsf{h}(\alpha_f) = \max\{\mathsf{m}(g), \mathsf{m}(h)\}$.*

Proof of the corollary. After the change of variables $x \mapsto y$, $y \mapsto x$, and $z \mapsto z^{-1}$, we can apply Theorem 5.9 to conclude that either $\mathsf{m}\bigl(g(\,\cdot\,,\eta)\bigr) > \mathsf{m}\bigl(h(\,\cdot\,,\eta)\bigr)$ for all $\eta \in \mathbb{S}$, or $\mathsf{m}\bigl(g(\,\cdot\,,\eta)\bigr) < \mathsf{m}\bigl(h(\,\cdot\,,\eta)\bigr)$ for all $\eta \in \mathbb{S}$. The result then follows from (8.9) by integrating over $\eta \in \mathbb{S}$. □

The intuition behind (8.9) is simple to explain. The matrices $A_{\zeta,f}(\xi,\eta)$ for $\zeta \in \Omega'_q$ take the form

$$A_{\zeta,f}(\xi,\eta) = \begin{bmatrix} g(\eta,\zeta) & h(\eta,\zeta)\xi & 0 & \dots & 0 \\ 0 & g(\eta\zeta,\zeta) & h(\eta\zeta,\zeta)\xi & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ h(\eta\zeta^{q-1},\zeta)\xi & 0 & \dots & \dots & g(\eta\zeta^{q-1},\zeta) \end{bmatrix}. \tag{8.10}$$

Then

$$\det A_{\zeta,f}(\xi,\eta) = \prod_{j=0}^{q-1} g(\eta\zeta^j,\zeta) + (-1)^{q-1}\xi^q \prod_{j=0}^{q-1} h(\eta\zeta^j,\zeta). \tag{8.11}$$

For fixed ζ the first product behaves like $\mathsf{M}\bigl(g(\,\cdot\,,\zeta)\bigr)^q$, and similarly for h. Whichever is larger will dominate, suggesting the formula (8.9).

To make this intuition precise, we require several lemmas.

Lemma 8.11. *For every $\xi \in \mathbb{C}$, $\zeta \in \mathbb{S}$, and $n \geqslant 1$,*

$$\frac{1}{n}\log|\xi^n - \zeta^n| \leqslant \log^+|\xi| + \frac{\log 2}{n}.$$

Proof. If $|\xi| \leqslant 1$, then $|\xi^n - \zeta^n| \leqslant 2$. If $|\xi| > 1$, then

$$\begin{aligned} \frac{1}{n}\log|\xi^n - \zeta^n| &\leqslant \frac{1}{n}\log(|\xi|^n + 1) \leqslant \frac{1}{n}\left\{\log|\xi|^n + \log\frac{|\xi|^n+1}{|\xi|^n}\right\} \\ &= \log|\xi| + \frac{1}{n}\log\left(1 + \frac{1}{|\xi|^n}\right) \leqslant \log|\xi| + \frac{\log 2}{n}. \end{aligned}$$

□

If $0 \neq \phi(u) \in \mathbb{C}[u]$, then (5.1) shows that $\mathsf{m}(\phi) > -\infty$. We will use the convention $\mathsf{m}(0) = -\infty$. With the usual conventions about arithmetic and about inequalities involving $-\infty$, the results that follow will make sense and are true even if some of the polynomials are 0.

Lemma 8.12. *Suppose that $\phi(u) \in \mathbb{C}[u]$ has degree $\leqslant D$. Then for every $\zeta \in \mathbb{S}$ and $n \geqslant 1$,*

$$R_n\big(\log|\phi|\big)(\zeta) := \frac{1}{n}\sum_{j=0}^{n-1} \log\big|\phi(e^{2\pi 2j/n}\zeta)\big| \leqslant \mathsf{m}(\phi) + \frac{D\log 2}{n}. \tag{8.12}$$

Proof. Let $n \geqslant 1$ and $\omega = e^{2\pi i/n}$, so that $R_n\big(\log|\phi|\big)(\zeta) = \frac{1}{n}\sum_{j=0}^{n-1}\log|\phi(\omega^j\zeta)|$. Let $\phi(u) = c_d u^d + \cdots + c_0$, where $d \leqslant D$ and $c_d \neq 0$. Then $\phi(u) = c_d\prod_{k=1}^{d}(u-\xi_k)$, and by Lemma 8.11,

$$\begin{aligned} R_n\big(\log|\phi|\big)(\zeta) &= \frac{1}{n}\sum_{j=0}^{n-1}\log\Big|c_d\prod_{k=1}^{d}(\omega^j\zeta-\xi_k)\Big| = \log|c_d| + \frac{1}{n}\sum_{k=1}^{d}\sum_{j=0}^{n-1}\log|\omega^j\zeta-\xi_k| \\ &= \log|c_d| + \frac{1}{n}\sum_{k=1}^{d}\log\Big|\prod_{j=0}^{n-1}(\omega^j\zeta-\xi_k)\Big| = \log|c_d| + \frac{1}{n}\sum_{k=1}^{d}\log|\zeta^n-\xi_k^n| \\ &\leqslant |c_d| + \sum_{k=1}^{d}\log^+|\xi_k| + \frac{d\log 2}{n} \leqslant \mathsf{m}(\phi) + \frac{D\log 2}{n}. \end{aligned}$$

□

Lemma 8.13. *Let $\phi(u), \psi(u) \in \mathbb{C}[u]$ each have degree $\leqslant D$. Then for every $n > (D\log 2)/\varepsilon$,*

$$\begin{aligned} \max\{\mathsf{m}(\phi),\mathsf{m}(\psi)\} &\leqslant \int_{\mathbb{S}} \max\big\{R_n\big(\log|\phi|\big)(\zeta), R_n\big(\log|\psi|\big)(\zeta)\big\}\,d\zeta \\ &\leqslant \max\{\mathsf{m}(\phi),\mathsf{m}(\psi)\} + \varepsilon. \end{aligned}$$

Proof. If $n > (D\log 2)/\varepsilon$, then the previous lemma implies that for every $\zeta \in \mathbb{S}$ we have $R_n\big(\log|\phi|\big)(\zeta) \leqslant \mathsf{m}(\phi) + \varepsilon$ and $R_n\big(\log|\psi|\big)(\zeta) \leqslant \mathsf{m}(\psi) + \varepsilon$. Hence,

$$\max\big\{R_n\big(\log|\phi|\big)(\zeta), R_n\big(\log|\psi|\big)(\zeta)\big\} \leqslant \max\{\mathsf{m}(\phi),\mathsf{m}(\psi)\} + \varepsilon,$$

and the second inequality follows by integrating over $\zeta \in \mathbb{S}$.

For the first inequality, observe that

$$\mathsf{m}(\phi) = \int_{\mathbb{S}} R_n\big(\log|\phi|\big)(\zeta)\,d\zeta \leqslant \int_{\mathbb{S}} \max\big\{R_n\big(\log|\phi|\big)(\zeta), R_n\big(\log|\psi|\big)(\zeta)\big\}\,d\zeta,$$

and similarly for ϕ. □

We need one more property of Mahler measure, proved by Boyd [10]. Recall that $\mathsf{M}(\phi) = \exp\big(\mathsf{m}(\phi)\big)$, and by convention $\exp(-\infty) = 0$.

Theorem 8.14 [10]. *The map $\mathbb{C}^{D+1} \to [0,\infty)$ given by*

$$(c_0, c_1, \ldots, c_D) \mapsto \mathsf{M}\big(c_0 + c_1 u + \cdots + c_D u^D\big)$$

is continuous.

Continuity is clear when the coefficients remain bounded away from 0 since the roots are continuous functions of the coefficients. But if $c_D \to 0$, for example, then continuity is more subtle. Boyd's proof, which also applies to polynomials in several variables with bounded degree, uses Graeffe's root-squaring method, thereby managing to sidestep various delicate issues and leading to a remarkably simple proof.

Proof of Theorem 8.9. If $\zeta \in \Omega'_q$, then as we saw in (8.11),

$$\det A_{\zeta,f}(\xi,\eta) = \prod_{j=0}^{q-1} g(\eta\zeta^j,\zeta) + (-1)^{q-1}\xi^q \prod_{j=0}^{q-1} h(\eta\zeta^j,\zeta).$$

By (5.1), for any complex numbers a and b,

$$\int_{\mathbb{S}} \log|a+\xi^q b|\,d\xi = \max\{\log|a|,\log|b|\}.$$

Hence,

$$\int_{\mathbb{S}} \log\bigl|\det A_{\zeta,f}(\xi,\eta)\bigr|\,d\xi = \max\biggl\{\log\biggl|\prod_{j=0}^{q-1} g(\eta\zeta^j,\zeta)\biggr|, \log\biggl|\prod_{j=0}^{q-1} h(\eta\zeta^j,\zeta)\biggr|\biggr\}.$$

It follows that

$$\begin{aligned}\frac{1}{q}\iint_{\mathbb{S}^2} &\log\bigl|\det A_{\zeta,f}(\xi,\eta)\bigr|\,d\xi\,d\eta \\ &= \int_{\mathbb{S}} \max\bigl\{R_q\bigl(\log|g(\,\cdot\,,\zeta)|\bigr)(\eta), R_q\bigl(\log|h(\,\cdot\,,\zeta)|\bigr)(\eta)\bigr\}\,d\eta.\end{aligned}$$

Let $\varepsilon > 0$. Writing $g_\zeta(y) = g(y,\zeta)$ and $\mathsf{h}_\zeta(y) = h(y,\zeta)$, we note that these are Laurent polynomials of uniformly bounded degree $\leqslant D$ in $\mathbb{C}[y^{\pm}]$, where the degree of a Laurent polynomial is the length of its Newton polygon. Applying Lemma 8.13 to g_ζ and h_ζ, we get that for $q > (D\log 2)/\varepsilon$

$$\begin{aligned}\max\bigl\{\mathsf{m}\bigl(g(\,\cdot\,,\zeta)\bigr), \mathsf{m}\bigl(h(\,\cdot\,,\zeta)\bigr)\bigr\} &\leqslant \frac{1}{q}\iint_{\mathbb{S}^2} \log\bigl|\det A_{\zeta,f}(\xi,\eta)\bigr|\,d\xi\,d\eta \\ &< \max\bigl\{\mathsf{m}\bigl(g(\,\cdot\,,\zeta)\bigr), \mathsf{m}\bigl(h(\,\cdot\,,\zeta)\bigr)\bigr\} + \varepsilon.\end{aligned}$$

We must now consider the possibility that for some $\zeta \in \mathbb{S}$ both $g(y,\zeta)$ and $h(y,\zeta)$ vanish as polynomials in $\mathbb{C}[y^{\pm}]$. We claim that this never happens if α_f is expansive. Let

$$g(y,z) = \sum_j g_j(z)y^j \quad\text{and}\quad h(y,z) = \sum_k h_k(z)y^k, \quad\text{where}\quad g_j(z), h_k(z) \in \mathbb{Z}[z^{\pm}].$$

If $\zeta_0 \in \mathbb{S}$ were a common zero for all the $g_j(z)$ and $h_k(z)$, then its minimal polynomial over $\mathbb{Q}$ would divide every $g_j(z)$ and $h_k(z)$. But then at the non-zero point $w = \sum_{i,j,k} \zeta_0^k\, x^i y^j z^k \in \ell^\infty(\Gamma,\mathbb{C})$ we would have $\rho_f(w) = 0$, contradicting expansiveness.

Thus, the function $\zeta \mapsto \max\{\mathsf{m}(g(\cdot,\zeta)), \mathsf{m}(h(\cdot,\zeta))\}$ is continuous on $\mathbb{S}$. Hence, the sums

$$\frac{1}{q^2} \sum_{\zeta \in \Omega_q} \iint_{\mathbb{S}^2} \log \left| \det A_{\zeta,f}(\xi,\eta) \right| d\xi \, d\eta$$

approximate $\mathsf{h}(\alpha_f)$ by Theorem 8.8 and at the same time are also Riemann sums over Ω_q of the continuous function $\max\{\mathsf{m}(g(\cdot,\zeta)), \mathsf{m}(h(\cdot,\zeta))\}$, and so converge to the integral in (8.9). □

Our proof of Theorem 8.9 made use of the expansiveness hypothesis for α_f to approximate the entropy with the help of the periodic points. But surely the entropy formula (8.9) is valid more generally.

Problem 8.15. Is the entropy formula (8.9), proven for linear polynomials in $\mathbb{Z}\Gamma$, valid for arbitrary $g(y,z)$ and $h(y,z)$ in $\mathbb{Z}[y^{\pm}, z^{\pm}]$?

It will follow from results in the next section that (8.9) is valid if either $g \equiv 1$ or $h \equiv 1$.

Let us turn to the quadratic case $f(x,y,z) = g_0(y,z) + xg_1(y,z) + x^2 g_2(y,z)$. We start with a simple result about determinants.

Lemma 8.16. *Let a_j, b_j, and c_j $(0 \leqslant j \leqslant q-1)$ be arbitrary complex numbers. Then*

$$\det \begin{bmatrix} a_0 & b_0 & c_0 & 0 & \dots & 0 & 0 \\ 0 & a_1 & b_1 & c_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ c_{q-2} & 0 & 0 & 0 & \dots & a_{q-2} & b_{q-2} \\ b_{q-1} & c_{q-1} & 0 & 0 & \dots & 0 & a_{q-1} \end{bmatrix} = \prod_{j=0}^{q-1} a_j - \operatorname{tr} \prod_{j=0}^{q-1} \begin{bmatrix} -b_j & c_j \\ -a_j & 0 \end{bmatrix} + \prod_{j=0}^{q-1} c_j.$$

If $c_j a_{j+1} = -b_j b_{j+1}$ for all j, where subscripts are taken mod q, then the value of this determinant simplifies to

$$\prod_{j=0}^{q-1} a_j - (-1)^q (\tau^q + \sigma^q) \prod_{j=0}^{q-1} b_j + \prod_{j=0}^{q-1} c_j,$$

where $\tau = (1+\sqrt{5})/2$ and $\sigma = -1/\tau$.

Proof. Taking subscripts mod q, a permutation π of $\{0,1,\dots,q-1\}$ contributes a non-zero summand in the expansion of the determinant if and only if it has the form $\pi(j) = j + \varepsilon_j$, where $\varepsilon_j = 0$, 1, or 2. The sequences $\{\varepsilon_j\} \in \{0,1,2\}^q$ corresponding to permutations are precisely the closed paths of length q in the labelled shift of finite type depicted in Fig. 2.

The paths $00\dots0$ and $22\dots2$ give the terms $a_0 a_1 \cdots a_{q-1}$ and $c_0 c_1 \cdots c_{q-1}$, respectively, while it is easy to check that the golden mean shift of finite type produces the middle term of the result.

If $c_j a_{j+1} = -b_j b_{j+1}$ for all j, then we can replace each occurrence of a block 20 in a closed path of length q by the block 11, changing the factor $c_j a_{j+1}$ to $b_j b_{j+1}$ together with an appropriate sign change. The result of these substitutions is that

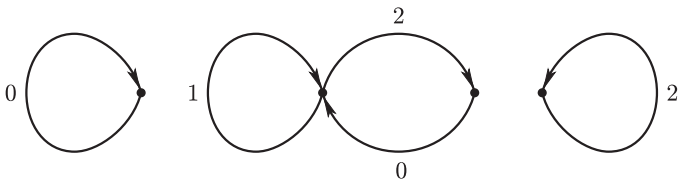

Figure 2

every closed path of length q in the golden mean shift gives the same contribution $-(-1)^q b_0 b_1 \cdots b_{q-1}$ to the expansion of the determinant, and the number of such paths is $\operatorname{tr}\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^q = \tau^q + \sigma^q$. □

We use this to derive quadratic analogues of (8.10) and (8.11).

Corollary 8.17. *Let $f(x,y,z) = g_0(y,z) + xg_1(y,z) + x^2 g_2(y,z) \in \mathbb{Z}\Gamma$. Suppose that $\zeta \in \Omega'_q$, and for $(\xi,\eta) \in \mathbb{S}^2$ define $A_{\zeta,f}(\xi,\eta)$ as above. Then*

$$\det A_{\zeta,f}(\xi,\eta) = \prod_{j=0}^{q-1} g_0(\eta\zeta^j,\zeta) + (-1)^{q-1} \operatorname{tr} \prod_{j=0}^{q-1} \begin{bmatrix} -g_1(\eta\zeta^j,\zeta)\xi & g_2(\eta\zeta^j,\zeta)\xi^2 \\ -g_0(\eta\zeta^j,\zeta) & 0 \end{bmatrix} + \xi^{2q} \prod_{j=0}^{q-1} g_2(\eta\zeta^j,\zeta). \tag{8.13}$$

If $g_1(y,z)g_1(yz,z) = -g_2(y,z)g_0(yz,z)$, then

$$\det A_{\zeta,f}(\xi,\eta) = \prod_{j=0}^{q-1} g_0(\eta\zeta^j,\zeta) + (-1)^{q-1}(\tau^q + \sigma^q)\xi^q \prod_{j=0}^{q-1} g_1(\eta\zeta^j,\zeta) + \xi^{2q} \prod_{j=0}^{q-1} g_2(\eta\zeta^j,\zeta). \tag{8.14}$$

Motivated by Theorem 8.8 and the rigorous results from the expansive linear case, we can now formulate a reasonable conjecture about entropy in the quadratic case.

As in the linear case, for fixed irrational ζ the growth rates of the first and third terms in (8.13) are given by $\mathsf{m}\big(g_0(\,\cdot\,,\zeta)\big)$ and $\mathsf{m}\big(g_2(\,\cdot\,,\zeta)\big)$. The growth rate for the second term should be the same for almost every choice of $(\xi,\eta) \in \mathbb{S}^2$, and if this is so, then denote this value by $b_f(\zeta)$. For example, in the case $g_1(y,z)g_1(yz,z) = -g_2(y,z)g_0(yz,z)$ we see from (8.14) that $b_f(\zeta) = \log\tau + \mathsf{m}\big(g_1(\,\cdot\,,\zeta)\big)$.

Problem 8.18. Let $f(x,y,z) = g_0(y,z) + xg_1(y,z) + x^2 g_2(y,z) \in \mathbb{Z}\Gamma$. Is

$$\mathsf{h}(\alpha_f) = \int_{\mathbb{S}} \max\big\{\mathsf{m}\big(g_0(\,\cdot\,,\zeta)\big), b_f(\zeta), \mathsf{m}\big(g_2(\,\cdot\,,\zeta)\big)\big\}\, d\zeta?$$

In particular, if

$$g_1(y,z)g_1(yz,z) = -g_2(y,z)g_0(yz,z),$$

is

$$\mathsf{h}(\alpha_f) = \int_{\mathbb{S}} \max\bigl\{\mathsf{m}\bigl(g_0(\,\cdot\,,\zeta)\bigr), \log\tau + \mathsf{m}\bigl(g_1(\,\cdot\,,\zeta)\bigr), \mathsf{m}\bigl(g_2(\,\cdot\,,\zeta)\bigr)\bigr\}\,d\zeta? \tag{8.15}$$

Example 8.19. Fix $g(y,z) \in \mathbb{Z}\Gamma$, and let $g_0(y,z) = -g(yz^{-1},z)g(y,z)$, $g_1(y,z) = g(y,z)$, and $g_2(y,z) = 1$. These satisfy the conditions for (8.15), and this formula becomes

$$\mathsf{h}(\alpha_f) = \int_{\mathbb{S}} \max\bigl\{2\mathsf{m}\bigl(g(\,\cdot\,,\zeta)\bigr), \log\tau + \mathsf{m}\bigl(g(\,\cdot\,,\zeta)\bigr), 0\bigr\}\,d\zeta. \tag{8.16}$$

For example, letting $g(y,z) = (z-1)y + z^2 - 1$, we see that for each of the three functions in (8.16) there is a range of ζ for which it is the maximum function.

A similar analysis can be carried out for higher-degree polynomials, but the evaluation of the relevant determinants now involves a finite family of more complicated (and interesting) shifts of finite type.

9. Lyapunov exponents and entropy

The methods of the previous section to compute entropy have some serious limitations because of the expansiveness assumptions. There is a more geometric approach to entropy, using the theory of Lyapunov exponents, which Deninger used in [16] to calculate Fuglede–Kadison determinants (or equivalently by Theorem 7.5, entropy) in much greater generality.

To motivate this approach, we first recall the linear example $f(x,y,z) = y - g(x,z) \in \mathbb{Z}\Gamma$. For $v_{\xi,\zeta}$ defined in (5.2) and for $w = \sum_{-\infty}^{\infty} c_n\, v_{\xi,\zeta}\, y^n$ with $\rho_f(w) = 0$, we have from (5.3) that

$$c_n = \left[\prod_{j=0}^{n-1} g(\xi\zeta^j,\zeta)\right] c_0. \tag{9.1}$$

For irrational ζ, the products in (9.1) have growth rate

$$\frac{1}{n}\log\left|\prod_{j=0}^{n-1} g(\xi\zeta^j,\zeta)\right| = \frac{1}{n}\sum_{j=0}^{n-1} \log|g(\xi\zeta^j,\zeta)| \to \mathsf{m}\bigl(g(\,\cdot\,,\zeta)\bigr),$$

and the limit is the same for almost every ξ by the ergodicity of irrational rotations. For toral automorphisms, the entropy is the sum of the growth rates on various eigenspaces that are positive. By analogy, we would expect the entropy here to be the integral of the positive growth rates, that is,

$$\mathsf{h}(\alpha_f) = \int_{\mathbb{S}} \max\bigl\{\mathsf{m}\bigl(g(\,\cdot\,,\zeta)\bigr), 0\bigr\}\,d\zeta. \tag{9.2}$$

This is a special case of (8.9), but with no assumptions on g.

Indeed, since the eigenspaces here are 1-dimensional, the techniques used in [44] can be adapted to prove the validity of (9.2) for every $0 \neq g \in \mathbb{Z}[x^{\pm}, z^{\pm}]$. However, since this will be subsumed under Deninger's results, there is no need to provide an independent proof here.

To state the main result in [16], we need to give a little background. For each irrational $\zeta \in \mathbb{S}$ there is the rotation algebra $\mathscr{A}_\zeta$, which is the von Neumann algebra version of the twisted l^1 algebras used in Allan's local principle (see [3] for details). There are also natural maps $\pi_\zeta \colon \mathscr{N}\Gamma \to \mathscr{A}_\zeta$. As explained in [15], § 5, there is a faithful normalized trace function tr_ζ on each $\mathscr{A}_\zeta$ such that $\mathrm{tr}_{\mathscr{N}\Gamma}(a) = \int_{\mathbb{S}} \mathrm{tr}_\zeta(a)\, d\zeta$ for every $a \in \mathscr{N}\Gamma$. This implies that for determinants we have

$$\log \det_{\mathscr{N}\Gamma}(a) = \int_{\mathbb{S}} \log \det\bigl(\pi_\zeta(a)\bigr)\, d\zeta. \tag{9.3}$$

Hence, we need a way of evaluating the integrands for $a = \rho_f$.

Suppose that $f \in \mathbb{Z}\Gamma$ is monic in y and of degree D, and so has the form

$$f(x, y, z) = y^D - g_{D-1}(x, z)y^{D-1} - \cdots - g_0(x, z),$$

where $g_j(x, z) \in \mathbb{Z}[x^{\pm}, z^{\pm}]$ and $g_0 \neq 0$. Calculations similar to those in Example 5.13 show that if $\rho_f\bigl(\sum_{-\infty}^{\infty} c_n v_{\xi,\zeta} y^n\bigr) = 0$, then for all n

$$c_{n+D} = g_{D-1}(\xi\zeta^n, \zeta)c_{n+D-1} + g_{D-2}(\xi\zeta^n, \zeta)c_{n+D-2} + \cdots + g_0(\xi\zeta^n, \zeta)c_n.$$

Put

$$A(\xi, \zeta) = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_0(\xi, \zeta) & g_1(\xi, \zeta) & g_2(\xi, \zeta) & \dots & g_{D-1}(\xi, \zeta) \end{bmatrix}, \tag{9.4}$$

and let

$$A_n(\xi, \zeta) = A(\xi\zeta^{n-1}, \zeta)A(\xi\zeta^{n-2}, \zeta) \cdots A(\xi, \zeta)$$

be the corresponding cocycle. Then the recurrence relation for $\{c_n\}$ can be written as

$$A_n(\xi, \zeta) \begin{bmatrix} c_0 \\ \vdots \\ c_{D-1} \end{bmatrix} = \begin{bmatrix} c_n \\ \vdots \\ c_{n+D-1} \end{bmatrix}.$$

The $A_n(\xi, \zeta)$ are $D \times D$ matrices, and we need to include all directions with positive growth rate. There is a deep and important theorem governing this.

Theorem 9.1 (Oseledets Multiplicative Ergodic Theorem [54]). *Let T be an invertible ergodic measure-preserving transformation on a probability space (Ω, ν), and let $B \colon \Omega \to \mathbb{C}^{D\times D}$ be a measurable map from Ω to the $D \times D$ complex matrices such that $\int_\Omega \log^+ \|B(\omega)\|\, d\nu(\omega) < \infty$. Then there exist a T-invariant measurable set $\Omega_0 \subset \Omega$ with $\nu(\Omega_0) = 1$, a number $M \leqslant D$, real numbers $\chi_1 < \chi_2 < \cdots < \chi_M$, multiplicities $r_1, \dots, r_M \geqslant 1$ with $r_1 + \cdots + r_M = D$, and measurable maps V_j from Ω_0 to the space of r_j-dimensional subspaces of $\mathbb{C}^D$ such that for all $\omega \in \Omega_0$*

1) $\mathbb{C}^D = V_1(\omega) \oplus \cdots \oplus V_M(\omega)$,
2) $B(\omega)V_j(\omega) = V_j(T\omega)$, *and*
3) $\dfrac{1}{n} \log \dfrac{\|B(T^{n-1}\omega)B(T^{n-2}\omega)\cdots B(\omega)v\|}{\|v\|} \to \chi_j$ *uniformly for* $0 \neq v \in V_j(\omega)$.

To apply this result to our situation, fix an irrational $\zeta \in \mathbb{S}$. Let $T_\zeta \colon \mathbb{S} \to \mathbb{S}$ be given by $T_\zeta(\xi) = \xi\zeta$, and let $B(\xi) = A(\xi, \zeta)$. Since the entries of the matrix $B(\xi)$ are continuous with respect to ξ, it is clear that $\int_{\mathbb{S}} \log^+ \|B(\xi)\| < \infty$. Hence, there are Lyapunov exponents $\chi_j(\zeta)$ and multiplicities $r_j(\zeta)$. With these in hand, we can now state Deninger's main result from [16] as it applies to the Heisenberg case.

Theorem 9.2. *Let* $f(x, y, z) = y^D - g_{D-1}(x, z)y^{D-1} - \cdots - g_0(x, z) \in \mathbb{Z}\Gamma$, *where* $g_j(x, z) \in \mathbb{Z}[x^{\pm}, z^{\pm}]$ *and* $g_0 \neq 0$. *For every irrational* ζ *denote the Lyapunov exponents for* $A(\xi, \zeta)$ *and* T_ζ *as above by* $\chi_j(\zeta)$ *with multiplicities* $r_j(\zeta)$. *Then*

$$\log \det_\zeta(\pi_\zeta(\rho_f)) = \sum_j r_j(\zeta)\chi_j(\zeta)^+$$

and hence by Theorem 7.5 and (9.3),

$$\mathsf{h}(\alpha_f) = \log \det\nolimits_{\mathcal{N}\Gamma} \rho_f = \int_{\mathbb{S}} \sum_j r_j(\zeta)\chi_j(\zeta)^+ \, d\zeta. \tag{9.5}$$

Example 9.3. Let $f(x, y, z) = y^2 - (2x - 1)y + 1$. For each $(\xi, \zeta) \in \mathbb{S}^2$ there is a non-zero vector $v(\xi, \zeta) \in \mathbb{C}^2$ and a multiplier $\kappa(\xi, \zeta) \in \mathbb{C}$ with $|\kappa(\xi, \zeta)| \geqslant 1$ such that

$$\begin{bmatrix} 0 & 1 \\ -1 & 2\xi - 1 \end{bmatrix} v(\xi, \zeta) = \kappa(\xi, \zeta)v(\xi\zeta, \zeta).$$

Since the determinants of these matrices all have absolute value 1, there is exactly one non-negative Lyapunov exponent of multiplicity 1. For each irrational $\zeta \in \mathbb{S}$ its value is given by $\chi(\zeta) = \int_{\mathbb{S}} \log|\kappa(\xi, \zeta)| \, d\xi$, and hence

$$\mathsf{h}(\alpha_f) = \int_{\mathbb{S}} \chi(\zeta) \, d\zeta = \iint_{\mathbb{S}^2} \log|\kappa(\xi, \zeta)| \, d\xi \, d\zeta.$$

A numerical calculation of the graph of $\log|\kappa(\xi, \zeta)|$ is shown in Fig. 3, and indicates the complexity of these phenomena even in the quadratic case.

Although Lyapunov exponents are generally difficult to compute, there is a method to obtain rigorous lower bounds on the largest Lyapunov exponent, known as 'Herman's subharmonic trick'. Its use in our context was suggested to us by Michael Björklund.

Proposition 9.4. *Let* $f(x, y, z) = y^D - g_{D-1}(x, z)y^{D-1} - \cdots - g_0(x, z) \in \mathbb{Z}\Gamma$, *where* $g_j(x, z) \in \mathbb{Z}[x, z^{\pm}]$, *so that only non-negative powers of* x *are allowed, and* $g_0(x, z) \neq 0$. *For every irrational* $\zeta \in \mathbb{S}$ *let* $\chi_\infty(\zeta)$ *denote the largest Lyapunov exponent in Theorem 9.2. Then*

$$\chi_\infty(\zeta) \geqslant \log \operatorname{spr} A(0, \zeta),$$

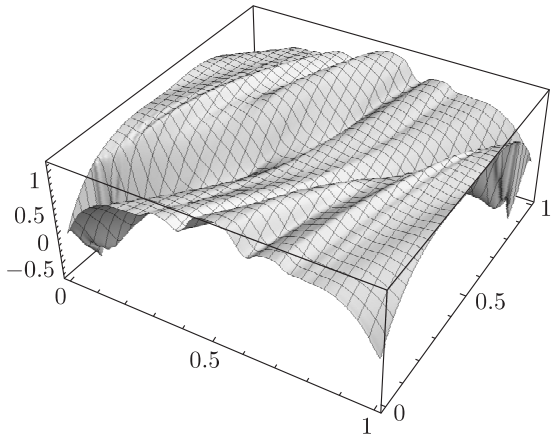

Figure 3. Graph of $\log|\kappa(\xi,\zeta)|$.

where $A(\xi,\zeta)$ *is the matrix given in* (9.4), *and* spr *denotes the spectral radius of a complex matrix. In particular*,

$$\mathsf{h}(\alpha_f) \geqslant \int_{\mathbb{S}} \log^+ \operatorname{spr} A(0,\zeta)\, d\zeta. \tag{9.6}$$

Proof. Fix an irrational $\zeta \in \mathbb{S}$. Put $B(x) = A(x,\zeta)$, and let $T(x) = \zeta x$. For a complex matrix $C = [c_{ij}]$ define $\|C\| = \max_{i,j}\{|c_{ij}|\}$. Theorem 9.1 shows that for almost every $\xi \in \mathbb{S}$ we have

$$\chi_\infty(\zeta) = \lim_{n\to\infty} \frac{1}{n} \log \|B(T^{n-1}\xi)B(T^{n-2}\xi)\cdots B(\xi)\|.$$

We multiply the matrices:

$$B(T^{n-1}x)B(T^{n-2}x)\cdots B(x) = [b_{ij}^{(n)}(x)],$$

where the $b_{ij}^{(n)}(x)$ are polynomials in x with complex coefficients. Each function $\log|b_{ij}^{(n)}(\xi)|$ is subharmonic for $\xi \in \mathbb{C}$, and hence $\max_{i,j}\{\log|b_{ij}^{(n)}(\xi)|\}$ is also subharmonic for $\xi \in \mathbb{C}$. Thus,

$$\max_{i,j}\bigl\{\log|b_{ij}^{(n)}(0)|\bigr\} \leqslant \int_{\mathbb{S}} \max\bigl\{\log|b_{ij}^{(n)}(\xi)|\bigr\}\, d\xi.$$

Furthermore,

$$\frac{1}{n}\max_{i,j}\bigl\{\log|b_{ij}^{(n)}(0)|\bigr\} = \frac{1}{n}\log\|B(0)^n\| \to \log\operatorname{spr} B(0) = \log\operatorname{spr} A(0,\zeta)$$

as $n \to \infty$. The entries in $A(\xi,\zeta)$ are uniformly bounded above, and hence $(1/n)\log|b_{ij}^{(n)}(\xi)|$ is uniformly bounded for all $n \geqslant 1$ and $\xi \in \mathbb{S}$. Thus,

$$\begin{aligned}\chi_\infty(\zeta) &= \int_{\mathbb{S}} \lim_{n\to\infty} \frac{1}{n} \log \|B(T^{n-1}\xi)B(T^{n-2}\xi)\cdots B(\xi)\|\, d\xi \\ &\geqslant \limsup_{n\to\infty} \int_{\mathbb{S}} \frac{1}{n} \log \|B(T^{n-1}\xi)B(T^{n-2}\xi)\cdots B(\xi)\|\, d\xi \\ &\geqslant \limsup_{n\to\infty} \frac{1}{n} \log \|B(0)^n\| = \log \operatorname{spr} A(0,\zeta). \qquad \square\end{aligned}$$

Observe that since only non-negative powers of x are allowed, this result is reminiscent of the face entropy inequality in Corollary 7.9. It is stronger, since it gives a lower bound for every irrational ζ, but the integrated form (9.6) is weaker, since it uses only the top Lyapunov exponent to give a lower bound for the entropy of the face corresponding to $x = 0$.

Example 9.5. We finish by returning to Example 5.13. Let $f(x,y,z) = y^2 - xy - 1 \in \mathbb{Z}\Gamma$. Using the change of variables $x \mapsto y$, $y \mapsto x$ and $z \mapsto z^{-1}$, f becomes monic and linear in y, hence by the previous theorem we can compute that $\mathsf{h}(\alpha_f) = 0$. On the other hand, treating f as monic and quadratic in y, we see that the Lyapunov exponents must all be $\leqslant 0$. But the determinant has absolute value 1, and so in fact the Lyapunov exponents must vanish almost everywhere. In other words,

$$\lim_{n\to\infty} \frac{1}{n} \log \left\| \prod_{j=n-1}^{0} \begin{bmatrix} 0 & 1 \\ 1 & \xi\zeta^j \end{bmatrix} \right\| = 0 \quad \text{for almost every } (\xi,\zeta) \in \mathbb{S}^2. \tag{9.7}$$

By taking transposes to reverse the order of the factors, we obtain (1.1) in the Introduction. Although this appears to be a simple result, we have not been able to obtain it as a consequence of any known results in random matrix theory.

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