Topological Entropy — Exercises

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Throughout G is a countable discrete group, and all actions are continuous.

1. Let θ be an irrational number in $[0, 1)$ and consider the rotation homeomorphism $T : \mathbb{T} \to \mathbb{T}$ given by $Tz = e^{2\pi i\theta}z$ for all $z \in \mathbb{T}$. Prove that T is minimal.

2. Prove that every homeomorphism $T : \mathbb{T} \to \mathbb{T}$ has zero topological entropy.

3. Let X be a compact metrizable space and $T : X \to X$ a homeomorphism. Let $n \in \mathbb{N}$. Prove that $h_{\text{top}}(T^n) = nh_{\text{top}}(T)$.

4. Let X be a compact metrizable space and $T : X \to X$ a homeomorphism. Prove that $h_{\text{top}}(T^{-1}) = h_{\text{top}}(T)$.

5. For every $r \in [0, \infty]$ provide an example, with justification, of a compact metrizable space X and a homeomorphism $T : X \to X$ such that $h_{\text{top}}(T) = r$.

6. Suppose that G is amenable. Let $G \curvearrowright X$ be an action on a compact metrizable space with compatible metric d. Given a finite set $F \subseteq G$ and $\varepsilon > 0$ we say that a set $A \subseteq X$ is (n, ε)-*spanning* if for every $x \in X$ there is a $y \in A$ such that $\max_{s \in F} d(sx, sy) < \varepsilon$. Establish a formula for the topological entropy $h_{\text{top}}(X, G)$ in terms of (n, ε)-spanning sets.

7. Suppose that G is amenable. Provide an example, with justification, of an action $G \curvearrowright X$ on the Cantor set with $h_{\text{top}}(X, G) = \infty$.

8. Suppose that G is sofic. Provide an example, with justification, of an action $G \curvearrowright X$ on the Cantor set with $h_\Sigma(X, G) = \infty$ for every sofic approximation sequence Σ.

9. Suppose that G is amenable. Prove that every profinite action $G \curvearrowright X$ on a compact metrizable space satisfies $h_{\text{top}}(X, G) = 0$.

10. Suppose that G is sofic. Prove that every profinite action $G \curvearrowright X$ on a compact metrizable space satisfies $h_\Sigma(X,G)=0$ for every sofic approximation sequence Σ.

11. Show that Følnerness for a sequence $\{F_n\}$ of nonempty finite subsets of G can be characterized by each of the following two properties:

    (a) $|\bigcap_{s\in K} s^{-1}F_n|/|F_n| \to 1$ as $n\to\infty$ for all finite sets $K\subseteq G$,

    (b) $|\{t\in F_n : Kt\subseteq F_n\}|/|F_n|\to 1$ as $n\to\infty$ for all finite sets $K\subseteq G$.

12. Show that if G is amenable then it is sofic.

13. Suppose that G is amenable with Følner sequence $\{F_n\}$. Let $G\curvearrowright X$ be an action on the Cantor set. Show that

$$h_{\text{top}}(X,G)=\sup_{\mathscr{P}}\lim_{n\to\infty}\frac{1}{|F_n|}\log\left|\mathscr{P}^{F_n}\right|$$

where the supremum is over all clopen partitions $\mathscr{P}$ of X and $\mathscr{P}^{F_n}$ denotes the collection of all sets of the form $\bigcap_{s\in F_n} s^{-1}A_s$ where each A_s is a member of $\mathscr{P}$.

14. Suppose that G is amenable. Show that the definition of topological entropy for actions of G on compact metrizable spaces does not depend on the choice of Følner sequence.

15. Suppose that G is amenable. Let $G\curvearrowright X$ and $G\curvearrowright Y$ be actions on compact metrizable spaces, and suppose that the second is a factor of the first, i.e., there is a continuous G-equivariant surjection from X onto Y. Show that $h_{\text{top}}(Y,G)\le h_{\text{top}}(X,G)$.

16. Let $d\in\mathbb{N}$ with $d\ge 2$ and let $\mathbb{Z}^d \overset{\alpha}{\curvearrowright} X$ be an action on a compact metrizable space with nonzero topological entropy. Show that the homeomorphism $T: X\to X$ defined by $Tx=\alpha_{(1,0,0,\dots,0)}x$ has infinite topological entropy.

17. Let G be a sofic group. Let $q\in\mathbb{N}$ and let $G\curvearrowright X$ be the restriction of the left shift action $G\curvearrowright\{1,\dots,q\}$ to a proper closed G-invariant set $X\subseteq\{1,\dots,q\}$. Show that $h_\Sigma(X,G)<\log$ for every sofic approximation sequence Σ. Use this to conclude that G is surjunctive.

18. Let $\Sigma=\{\sigma_i : G\to \mathrm{Sym}(D_i)\}$ and $\Omega=\{\omega_i : G\to\mathrm{Sym}(D_i)\}$ be sofic approximation sequences which agree asymptotically in the sense that

$$\lim_{i\to\infty}\frac{1}{|D_i|}|\{v\in D_i : \sigma_{i,s}v=\omega_{i,s}v\}|=1$$

for all $s \in G$. Show that $h_\Sigma(X, G) = h_\Omega(X, G)$ for every action $G \curvearrowright X$ on a compact metrizable space.

19. Suppose that G is amenable. Let $G \curvearrowright X$ and $G \curvearrowright Y$ be actions on a compact metrizable spaces, and consider the product action $G \curvearrowright X \times Y$ defined by $s(x, y) = (sx, sy)$. Show that $h_{\text{top}}(X \times Y, G) = h_{\text{top}}(X, G) + h_{\text{top}}(Y, G)$.

20. Suppose that G is sofic with sofic approximation sequence Σ. Let $G \curvearrowright X$ and $G \curvearrowright Y$ be actions on a compact metrizable spaces, and consider the product action $G \curvearrowright X \times Y$ defined by $s(x, y) = (sx, sy)$. Show that $h_\Sigma(X \times Y, G) \leq h_\Sigma(X, G) + h_\Sigma(Y, G)$.

21. Suppose that G is amenable. Prove that every action $G \curvearrowright X$ on a compact metrizable space has a largest factor $G \curvearrowright Y$ with $h_{\text{top}}(Y, G) = 0$, meaning that every factor map $X \to Z$ onto an action $G \curvearrowright Z$ with $h_{\text{top}}(Z, G) = 0$ can be written as a composition $X \to Y \to Z$ of factor maps.

22. Suppose that G is sofic with sofic approximation sequence Σ. Prove that every action $G \curvearrowright X$ on a compact metrizable space has a largest factor $G \curvearrowright Y$ with $h_\Sigma(Y, G) \in \{0, -\infty\}$, meaning that every factor map $X \to Z$ onto an action $G \curvearrowright Z$ with $h_{\text{top}}(Z, G) \in \{0, -\infty\}$ can be written as a composition $X \to Y \to Z$ of factor maps.