Throughout $G$ is a countable discrete group, and all actions are continuous.

1. Let $\theta$ be an irrational number in $[0, 1)$ and consider the rotation homeomorphism $T : \mathbb{T} \to \mathbb{T}$ given by $Tz = e^{2\pi i \theta} z$ for all $z \in \mathbb{T}$. Prove that $T$ is minimal.

2. Prove that every homeomorphism $T : \mathbb{T} \to \mathbb{T}$ has zero topological entropy.

3. Let $X$ be a compact metrizable space and $T : X \to X$ a homeomorphism. Let $n \in \mathbb{N}$. Prove that $h_{\text{top}}(T^n) = nh_{\text{top}}(T)$.

4. Let $X$ be a compact metrizable space and $T : X \to X$ a homeomorphism. Prove that $h_{\text{top}}(T^{-1}) = h_{\text{top}}(T)$.

5. For every $r \in [0, \infty]$ provide an example, with justification, of a compact metrizable space $X$ and a homeomorphism $T : X \to X$ such that $h_{\text{top}}(T) = r$.

6. Suppose that $G$ is amenable. Let $G \curvearrowright X$ be an action on a compact metrizable space with compatible metric $d$. Given a finite set $F \subseteq G$ and $\varepsilon > 0$ we say that a set $A \subseteq X$ is $(n, \varepsilon)$-spanning if for every $x \in X$ there is a $y \in A$ such that $\max_{s \in F} d(sx, sy) < \varepsilon$. Establish a formula for the topological entropy $h_{\text{top}}(X, G)$ in terms of $(n, \varepsilon)$-spanning sets.

7. Suppose that $G$ is amenable. Provide an example, with justification, of an action $G \curvearrowright X$ on the Cantor set with $h_{\text{top}}(X, G) = \infty$.

8. Suppose that $G$ is sofic. Provide an example, with justification, of an action $G \curvearrowright X$ on the Cantor set with $h_{\Sigma}(X, G) = \infty$ for every sofic approximation sequence $\Sigma$.

9. Suppose that $G$ is amenable. Prove that every profinite action $G \curvearrowright X$ on a compact metrizable space satisfies $h_{\text{top}}(X, G) = 0$. 

10. Suppose that $G$ is sofic. Prove that every profinite action $G \curvearrowright X$ on a compact metrizable space satisfies $h_\Sigma(X, G) = 0$ for every sofic approximation sequence $\Sigma$.

11. Show that Følnerness for a sequence $\{F_n\}$ of nonempty finite subsets of $G$ can be characterized by each of the following two properties:
   
   (a) $\bigcap_{s \in K} s^{-1}F_n \big/ |F_n| \rightarrow 1$ as $n \rightarrow \infty$ for all finite sets $K \subseteq G$,
   
   (b) $|\{t \in F_n : Kt \subseteq F_n\}| / |F_n| \rightarrow 1$ as $n \rightarrow \infty$ for all finite sets $K \subseteq G$.

12. Show that if $G$ is amenable then it is sofic.

13. Suppose that $G$ is amenable with Følner sequence $\{F_n\}$. Let $G \curvearrowright X$ be an action on the Cantor set. Show that
   
   $$h_{\text{top}}(X, G) = \sup_{\mathcal{P}} \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \log |\mathcal{P}^F_n|$$
   
   where the supremum is over all clopen partitions $\mathcal{P}$ of $X$ and $\mathcal{P}^F_n$ denotes the collection of all sets of the form $\bigcap_{s \in F_n} s^{-1}A_s$ where each $A_s$ is a member of $\mathcal{P}$.

14. Suppose that $G$ is amenable. Show that the definition of topological entropy for actions of $G$ on compact metrizable spaces does not depend on the choice of Følner sequence.

15. Suppose that $G$ is amenable. Let $G \curvearrowright X$ and $G \curvearrowright Y$ be actions on compact metrizable spaces, and suppose that the second is a factor of the first, i.e., there is a continuous $G$-equivariant surjection from $X$ onto $Y$. Show that $h_{\text{top}}(Y, G) \leq h_{\text{top}}(X, G)$.

16. Let $d \in \mathbb{N}$ with $d \geq 2$ and let $\mathbb{Z}^d \curvearrowright X$ be an action on a compact metrizable space with nonzero topological entropy. Show that the homeomorphism $T : X \rightarrow X$ defined by $Tx = \alpha_{(1, 0, 0, \ldots, 0)} x$ has infinite topological entropy.

17. Let $G$ be a sofic group. Let $q \in \mathbb{N}$ and let $G \curvearrowright X$ be the restriction of the left shift action $G \curvearrowright \{1, \ldots, q\}$ to a proper closed $G$-invariant set $X \subseteq \{1, \ldots, q\}$. Show that $h_\Sigma(X, G) < \log q$ for every sofic approximation sequence $\Sigma$. Use this to conclude that $G$ is surjunctive.

18. Let $\Sigma = \{\sigma_i : G \rightarrow \text{Sym}(D_i)\}$ and $\Omega = \{\omega_i : G \rightarrow \text{Sym}(D_i)\}$ be sofic approximation sequences which agree asymptotically in the sense that
   
   $$\lim_{i \rightarrow \infty} \frac{1}{|D_i|} |\{v \in D_i : \sigma_{i,v} = \omega_{i,v}\}| = 1$$
for all $s \in G$. Show that $h_{\Sigma}(X, G) = h_{\Omega}(X, G)$ for every action $G \curvearrowright X$ on a compact metrizable space.

19. Suppose that $G$ is amenable. Let $G \curvearrowright X$ and $G \curvearrowright Y$ be actions on a compact metrizable spaces, and consider the product action $G \curvearrowright X \times Y$ defined by $s(x, y) = (sx, sy)$. Show that $h_{\text{top}}(X \times Y, G) = h_{\text{top}}(X, G) + h_{\text{top}}(Y, G)$.

20. Suppose that $G$ is sofic with sofic approximation sequence $\Sigma$. Let $G \curvearrowright X$ and $G \curvearrowright Y$ be actions on a compact metrizable spaces, and consider the product action $G \curvearrowright X \times Y$ defined by $s(x, y) = (sx, sy)$. Show that $h_{\Sigma}(X \times Y, G) \leq h_{\Sigma}(X, G) + h_{\Sigma}(Y, G)$.

21. Suppose that $G$ is amenable. Prove that every action $G \curvearrowright X$ on a compact metrizable space has a largest factor $G \curvearrowright Y$ with $h_{\text{top}}(Y, G) = 0$, meaning that every factor map $X \to Z$ onto an action $G \curvearrowright Z$ with $h_{\text{top}}(Z, G) = 0$ can be written as a composition $X \to Y \to Z$ of factor maps.

22. Suppose that $G$ is sofic with sofic approximation sequence $\Sigma$. Prove that every action $G \curvearrowright X$ on a compact metrizable space has a largest factor $G \curvearrowright Y$ with $h_{\Sigma}(Y, G) \in \{0, -\infty\}$, meaning that every factor map $X \to Z$ onto an action $G \curvearrowright Z$ with $h_{\text{top}}(Z, G) \in \{0, -\infty\}$ can be written as a composition $X \to Y \to Z$ of factor maps.