# Homework exercises

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**Abstract**

Please attempt at least one problem per set. The sets are intended to correspond to lectures.

## 1 Set #1

1. Prove the rank 2 free group $\langle a, b\rangle$ is non-amenable.

2. The Ornstein Isomorphism Theorem states that if two Bernoulli shifts over the integers have the same base entropy then they are isomorphic. That is, if $H(\mu) = H(\nu)$ where $\mu \in \mathrm{Prob}(A), \nu \in \mathrm{Prob}(B)$ then the Bernoulli shifts $\mathbb{Z} \curvearrowright (A^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ and $\mathbb{Z} \curvearrowright (B^{\mathbb{Z}}, \nu^{\mathbb{Z}})$ are measurably conjugate. Prove that the same statement is true for the free group $\mathbb{F}_2 := \langle a, b\rangle$. Hint: identify $\mathbb{Z}$ with a subgroup of $\mathbb{F}_2$. You can build the isomorphism $(A^{\mathbb{F}_2}, \mu^{\mathbb{F}_2}) \to (B^{\mathbb{F}_2}, \nu^{\mathbb{F}_2})$ from an isomorphism for $(A^{\mathbb{Z}}, \mu^{\mathbb{Z}}) \to (B^{\mathbb{Z}}, \nu^{\mathbb{Z}})$ coset-by-coset.

## 2 Set #2

1. A point $x \in A^{\mathbb{Z}}$ is **periodic** if there is some integer $n$ such that $x_i = x_{i+n}$ for all $i$. If $x$ is periodic with period $n$ then let $\mu_x := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x}$ be the uniform measure on its

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orbit. We call such a measure a **periodic** measure. Prove that the periodic measures are dense in $\mathrm{Prob}_{\mathbb{Z}}(A^{\mathbb{Z}})$.

2. Prove that the rank 2 free group $\mathbb{F}_2$ is residually finite. This means that for every $g \in \mathbb{F}_2$ that is nontrivial there exists a homomorphism $\pi : \mathbb{F}_2 \to Q$ (where $Q$ is finite) such that $\pi(g) \neq 1$. Hint: it might help to think of $\mathbb{F}_2$ as the fundamental group of a figure 8.

# 3 Set #3

Let $\Gamma \curvearrowright (X, \mu)$ be a measure-preserving action on a standard probability space. A sequence $\{A_n\}$ of Borel subsets of $X$ is **asymptotically invariant** if $\mu(A_n \vartriangle gA_n) = 0$ for every $g \in \Gamma$. It is **nontrivial** if $\liminf_{n\to\infty} \mu(A_n) > 0$ and $\limsup_{n\to\infty} \mu(A_n) < 1$. The action $\Gamma \curvearrowright (X, \mu)$ is called **strongly ergodic** if there does not exist a nontrivial asymptotically invariant sequence.

1. Suppose $\Sigma$ is a sofic approximation to $\Gamma$ by expanders and $h_\Sigma(\mu) \geq 0$ for some measure $\mu \in \mathrm{Prob}_\Gamma(A^\Gamma)$. Prove that $\Gamma \curvearrowright (X, \mu)$ is strongly ergodic.

2. Prove that if $\Gamma$ is amenable and infinite then any strongly ergodic action of $\Gamma$ is an action on a finite set (up to measure 0).

3. Prove that if $\Gamma$ is non-amenable then every Bernoulli action of $\Gamma$ is strongly ergodic.

# 4 Set #4

1. Prove amenable groups are sofic.

2. Prove that if $\Gamma, \Lambda$ are sofic groups then the free group $\Gamma * \Lambda$ is also sofic. Hint: let $\sigma_n : \Gamma \to \mathrm{sym}(V_n), \sigma'_n : \Lambda \to \mathrm{sym}(V_n)$ be "good" maps. Let $\pi$ be a uniformly random permutation in $\mathrm{sym}(n)$. Show that the map $\sigma''_n : \Gamma * \Lambda \to \mathrm{sym}(V_n)$ given by

$$\sigma''_n(\gamma_1\lambda_1 \cdots \gamma_k\lambda_k) := \sigma_n(\gamma_1)(\pi\sigma'_n(\lambda_1)\pi^{-1}) \cdots \sigma_n(\gamma_k)\pi\sigma'_n(\lambda_k)\pi^{-1}$$

(if $\gamma_1, \ldots, \gamma_k \in \Gamma, \lambda_1, \ldots, \lambda_k \in \Lambda$ are nontrivial) is a "good" map of the free product.

## 5 Set #5

1. Prove that the $f$-invariant is additive under direct products. In other words, $f(\mu \times \nu) = f(\mu) + f(\nu)$ whenever $\mu \in \mathrm{Prob}_\Gamma(A^\Gamma), \nu \in \mathrm{Prob}(B^\Gamma)$, and $\Gamma$ is a free group.

2. Prove that sofic entropy is sub-additive under direct products. That is, $h_\Sigma(\mu \times \nu) \leq h_\Sigma(\mu) + h_\Sigma(\nu)$.